King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 550 Linear Algebra (Term 211)

Final Exam (Duration = 3 hours)

Problem 1. Let A be a 7×7 complex matrix in *rational form* that has *two* distinct characteristic values and *four* invariant factors, and such that $A^3 + A^2 = A + I$. Let f denote the characteristic polynomial of A.

(1) Assume A is *diagonalizable* and $f(0) > 0$. Find A and its invariant factors.

(2) Assume A is *NOT diagonalizable* and $f(0) < 0$. Find A and its invariant factors.

Problem 2. Let $A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 2 & 1 \end{pmatrix}$.

(1) Reduce $xI - A$ to its Smith normal form.

(2) Find the Jordan form J of A.

(3) Let T be a linear operator on $\mathbb{R}^3$ such that A is the matrix associated to T in the standard basis $\{e_1, e_2, e_3\}$. Find the respective T-annihilators of e_1, e_2, and e_3.

(4) Show that T has a cyclic vector; namely, find $\alpha \in \mathbb{R}^3$ such that $\mathbb{R}^3 = Z(\alpha, T)$, and give the matrix of T in the basis $S := \{\alpha, T\alpha, T^2\alpha\}$

Problem 3. Consider the basis $S := \left\{\beta_1 := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \beta_2 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \beta_3 := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$ in $\mathbb{R}^3$ equipped with the standard inner product.

(1) Apply the Gram-Schmidt process to S to obtain an *orthonormal* basis $B := \{\alpha_1, \alpha_2, \alpha_3\}$

(2) Express an arbitary vector $\alpha := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.

(3) Find the matrix G of the inner product in both bases S and B.

Problem 4. Let V be a finite-dimensional vector space over $F \subseteq \mathbb{C}$ and let L_1 and L_2 be two *nonzero* linear functionals on V. Consider the bilinear form

$$f(\alpha,\beta) = L_1\alpha\, L_2\beta \;-\; L_1\beta\, L_2\alpha \;,\; \forall\, \alpha,\beta \in V$$

(1) Show that L_1 and L_2 are linearly dependent $\Longleftrightarrow$ $f = 0$

Next, let $V = \mathbb{R}^3$ and let

$$L_1: \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & x+y \end{array} \quad ; \quad L_2: \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & y+z \end{array}$$

(2) Find the matrix of f in the standard ordered basis $S := \left\{e_1, e_2, e_3\right\}$ and find the rank of f.

(3) Find an ordered basis $B := \left\{\alpha_1, \alpha_2, \alpha_3\right\}$ such that the matrix of f in B is

$$\left[f\right]_B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Problem 5. Let V be a finite-dimensional vector space over F (= $\mathbb{R}$ or $\mathbb{C}$). Let W be a subspace of V, so that $V = W \oplus W^{\perp}$ (i.e., each $\alpha \in V$ is uniquely expressed in the form $\alpha = \beta + \gamma$ with $\beta \in W$ and $\gamma \in W^{\perp}$). Consider the linear operator

$$T: \begin{array}{ccc} V = W \oplus W^{\perp} & \longrightarrow & V \\ \alpha = \beta + \gamma & \mapsto & \beta - \gamma \end{array}$$

(1) Let E be *the orthogonal projection of V on W*. Express T in terms of E; namely, find $a, b \in F$ such that $T = aE + bI$.

(2) Use (1) to show that T is *self-adjoint* and *unitary*.

(3) Next, let $V = \mathbb{R}^3$, with standard inner product, and let W be the subspace of $\mathbb{R}^3$ spanned by the vector $e := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Find E.

(4) Find the matrix of T in the standard ordered basis $S := \left\{e_1, e_2, e_3\right\}$ of $\mathbb{R}^3$.

Problem 6. Let V be a finite-dimensional complex inner product space of dimension n and let T be a linear operator on V.

(1) Use induction on n to prove that there is an orthonormal basis $B := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for V such that the matrix $[T]_B$ is *upper triangular*.

(2) Let $A := [T]_B = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ & & \cdot & \cdot & \cdot & \cdot \\ & 0 & & \ddots & \cdot & \cdot \\ & & & & & \cdot \\ & & & & & a_{nn} \end{pmatrix}$.

Prove that if T is *normal*, then A is *diagonal*.