King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 212

Exam 2 : Math 550, Linear Algebra

Duration: 3 Hours

NAME :

ID :

Exercise 1. (5-5-5)

Let $V = \mathcal{M}_n(\mathbb{F})$ be the vector space of $n \times n$ matrices with coefficients in the field F, D a diagonal matrix and U be the linear operator on V defined by $U(B) = DB - BD$ for every $B \in V$.

(1) Prove that U is diagonalizable.

(2) Let A be a fixed (but arbitrary) matrix in V and define the linear operator T_A on V by setting $T_A(B) = AB - BA$ for every $B \in V$. Prove that if A is diagonalizable, then T_A is diagonalizable.

(3) Let $\mathcal{F} = \{T_A | A \text{ is diagonal}\}$. Prove that $\mathcal{F}$ is simultaneousely diagonalizable.

Exercise 2. (6-6-6-4-3 points)

Let $V = \mathbb{R}^4$ and T be the linear operator on V defined by:

$T(x, y, z, t) = (2x + y + z + 2t, 2y + t, 2z - t, t)$. Use the standard basis to:

(1) Find the Smith normal form of $xI - T$ and the invariant factors of T.

(2) Find the cyclic decomposition of $\mathbb{R}^4$ under T.

(3) Find the primary decomposition of $\mathbb{R}^4$ under T.

(4) Find the rational matrix form of T.

(5) Find the Jordan matrix form of T.

Exercise 3. (5-5-5-5)

Let V be an n-dimensional vector space over a field F, T a linear operator on V with minimal polynomial $P = P_1^{r_1} \dots P_k^{r_k}$, where P_i are irreducible monic polynomials.

(1) Prove that there are projections $E_1 \dots, E_k$ such that $V = W_1 \bigoplus \cdots \bigoplus W_k$ with $W_i = range(E_i)$.

Application: Assume that $V = \mathbb{R}^3$ as a vector space over $\mathbb{R}$, and let $M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ be the matrix of T in the standard basis.

(2) Find the minimal and characterestic polynomials of T.

(3) Find explicitely the corresponding projections E_i, $i = 1 \ldots, k$ (in matrix forms).

(4) Find a cyclic vector of T (if it exists).

Exercise 4. (5-3-2 points)

Let V be an n-dimensional complex vector space and T be a linear operator on V.

(1) Prove that there is a sequence of subspaces $V_1 \subsetneqq V_2 \subsetneqq \cdots \subsetneqq V_n = V$ such that $dim(V_i) = i$, and V_i is T-invariant for every $i = 1, \ldots, n$.

Assume that $V = \mathbb{C}^3$ as a vector space over $\mathbb{C}$, S its standard basis and T the linear operator given by $T(x, y, z) = (x + iz, x + iy + z, ix + 3z)$.

(2) Find a sequence $V_1 \subsetneqq V_2 \subsetneqq V_3 = V$ of T as in question (1).

(3) Find a basis B of V where the matrix $[T]_B$ representing T is upper triangular.

Exercise 5. (5-5-5-5)

Let V be an n-dimensional vector space over a field $\mathbb{F}$ with a scalar product $(|)$ that is not necessarily definite positive. Let $B = \{u_1, \ldots, u_n\}$ and $B' = \{v_1, \ldots, v_n\}$ to arbitrary orthogonal bases of V and assume that:

$(u_i|u_i) > 0$ for $i = 1, \ldots, r$, $(u_i|u_i) < 0$ for $i = r + 1, \ldots, s$ and $(u_i|u_i) = 0$ for $i = s + 1, \ldots, n$. Also $(v_i|v_i) > 0$ for $i = 1, \ldots, p$, $(v_i|v_i) < 0$ for $i = p + 1, \ldots, q$ and $(u_i|u_i) = 0$ for $i = q + 1, \ldots, n$.

(1) Prove that $\{u_1, \ldots, u_r, v_{p+1}, \ldots, v_n\}$ are linearly independent.

(2) Prove that $\{v_1, \ldots, v_p, u_{r+1}, \ldots, u_n\}$ are linearly independent.

(3) Prove that $r = p$. This common number for all orthogonal bases is called the index of positivity.

(4) Let $V = \mathbb{R}^4$ with the scalar product defined by $(X|Y) = xx' + yy' - zz' + tt'$ for every $X = (x, y, z, t)$ and $Y = (x', y', z', t')$. Find an orthogonal basis of V and its index of positivity.