King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 212

Final Exam : Math 550, Linear Algebra

Duration: 4 Hours

NAME :

ID :

Exercise 1. (5-5-5)

Let V be an n-dimensional vector space over a field F and T be a linear operator on V.

(1) Assume that $rank(T) = 1$. Show that V has a basis B such that

$$[T]_B = \begin{pmatrix} 0 & 0 & \dots & 0 & c_1 \\ 0 & 0 & \dots & 0 & c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & c_n \end{pmatrix}$$

(2) Prove that either $T^2 = 0$ or T is diagonalizable (Hint: separate the cases where $c_n = 0$ or $c_n \neq 0$).

(3) Assume that $T^n = 0$ and $T^{n-1} \neq 0$. Prove that there is a basis B' ov V such that

$$[T]_{B'} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

Exercise 2. (6-6-4-4 points)

Let A be a 7×7 complex matrix in *rational form* that has *two* distinct characteristic values and *four* invariant factors, and such that $A^3 + A^2 = A + I$. Let f denote the characteristic polynomial of A.

(1) Assume A is *diagonalizable* and $f(0) > 0$. Find A and its invariant factors.

(2) Assume A is *NOT diagonalizable* and $f(0) < 0$. Find A and its invariant factors.

(3) Prove that there is no real matrix A of odd order with minimal polynomial $P_0 = X^2 - 2X + 2$.

(4) Give an example of a complex 3×3 complex matrix with minimal polynomial $P_0 = X^2 - 2X + 2$.

Exercise 3. (5-5-6-4 points)

Let $A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 2 & 1 \end{pmatrix}$.

(1) Reduce $xI - A$ to its Smith normal form.

(2) Find the Jordan form J of A.

(3) Let T be a linear operator on $\mathbb{R}^3$ such that A is the matrix associated to T in the standard basis $\{e_1, e_2, e_3\}$. Find the respective T-annihilators of e_1, e_2, and e_3.

(4) Show that T has a cyclic vector; namely, find $\alpha \in \mathbb{R}^3$ such that $\mathbb{R}^3 = \mathrm{Z}(\alpha, T)$, and give the matrix of T in the basis $S := \{\alpha, T\alpha, T^2\alpha\}$

Exercise 4. (4-3-3-5 points)

Let V be a complex inner product space (not necessarily finite-dimensional) and T a linear operator on V which admits an adjoint T^*.

(1) Prove that for every $x, y \in V$,

$(Tx|y) = \frac{1}{4}[(T(x+y)|x+y) - (T(x-y)|x-y) + i(T(x+iy)|x+iy) - i(T(x-iy)|x-iy)]$.

(2) Prove that if T satisfies $(Tx|x) = 0$ for all $x \in V$, then $T = 0$ (Hint: use (1))

(3) Prove that if T is self-adjoint, then $(Tx|x)$ is real for all $x \in V$.

(4) Prove that if $(Tx|x)$ is real for all $x \in V$, then $T = T^*$ [Hint: Apply (2) to the operator $U = T - T^*$].

Exercise 5. (5-5-5-10-5 points)

Let V be an n-dimensional complex inner product space and T a linear operator on V.

A/ Assume that T is normal.

(1) Prove that c is a characterestic value of T if and only if $\bar{c}$ is a characterestic value of T^*.

(2) Prove that characterestic vectors associated to distinct characterestic values are orthogonal.

(3) Prove that there are commutating self-adjoints operators T_1 and T_2 such that $T = T_1 + iT_2$.

(4) Prove that there is a polynomial $f(X) \in \mathbb{C}[X]$ such that $T^* = f(T)$.

B/ Assume that T is a unitary operator.

(5) Prove that if c is a characterestic value of T, then $|c| = 1$.

Exercise 6. (8-6-6)

Let V be a finite-dimensional vector space over $F \subseteq \mathbb{C}$ and let L_1 and L_2 be two *nonzero* linear functionals on V. Consider the bilinear form

$$f(\alpha, \beta) = L_1\alpha \; L_2\beta \; - \; L_1\beta \; L_2\alpha \; , \; \forall \; \alpha, \beta \in V$$

(1) Show that L_1 and L_2 are linearly dependent $\Longleftrightarrow$ $f = 0$.

Next, let $V = \mathbb{R}^3$ and let

$$L_1 : \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & x + y \end{array} \quad ; \quad L_2 : \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & y + z \end{array}$$

(2) Find the matrix of f in the standard ordered basis $S := \{e_1, e_2, e_3\}$ and find the rank of f.

(3) Find an ordered basis $B := \{\alpha_1, \alpha_2, \alpha_3\}$ such that the matrix of f in B is

$$[f]_B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$