King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 221

Exam 1 : Math 550, Linear Algebra

Duration: 3 Hours

NAME :

ID :

Exercise 1. (5-6-3-5-3)

Let V be an n-dimensional vector space over a field F, V^* its dual space.

(1) Prove that $dim(V) = dim(V^*)$.

Let $V = \mathbb{R}^3$, V^* its dual space, $B = \{u_1 = (1,1,1), u_2 = (1,0,1), u_3 = (1,0,0)\}$ a basis of V, $B^* = \{u_1^*, u_2^*, u_3^*\}$ its dual basis and T the linear operator on V defined by $T(x,y,z) = (y-z, z-x, x-y)$.

(2) Find the matrix $[T]_B$ representing T in the basis B.

(3) Is T a nonsingular operator? Justify?

(4) Find explicitely B^*.

(5) Find $[T^t]_{B^*}$ where T^t is the operator transpose of T.

Exercise 2. (4-4-5-5 points)

Let $V = \mathcal{M}_n(\mathbb{R})$ be the real vector space of all $n \times n$ matrices with coefficients in $\mathbb{R}$, $\{A_{ij}\}_{1 \leq i,j \leq n}$ its standard basis where A_{ij} takes 1 in the ith row and jth column and zero elsewhere, V^* the dual space of V and consider the following subspaces of V and V^* respectively:

$W_1 = \{A \in V | AB = BA \text{ for every } B \in V\}$;

$W_2 = \{f \in V^* | f(A_{11}) + f(A_{22}) + \cdots + f(A_{nn}) = 0\}$.

(1) Find AA_{ij} and $A_{ij}A$ for every $A \in V$ and i, j.

(2) If D is a diagonal matrix of V, find DA_{i1} and $A_{i1}D$ for every $i \geq 2$.

(3) Use (1) and (2) to prove that $W_1 = \{aI_n | a \in \mathbb{R}\}$, I_n is the identity matrix.

(4) Prove that W_2 is a hyperspace of V^*.

Exercise 3. (6-6-2-6-2-3 points)

Let V be a vector space over a field $\mathbb{F}$ and T, H linear operators on V.

(1) Assume that $T = T^2$. Prove that $V = Ker(T) \bigoplus Im(T)$.

(2) Assume that $dim_{\mathbb{F}}(V) = n$ and suppose that $H^n = 0$ and $H^{n-1} \neq 0$. Prove that there is a basis B of V such that $[H]_B = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}$

Applications: Assume that $V = \mathbb{R}^3$

(3) Set $T(x,y,z) = (x+y, 0, x+y)$. Verify that $T = T^2$.

(4) Find $Ker(T)$, $Im(T)$, their bases and their dimensions.

(5) Set $H(x,y,z) = (y+z, z, 0)$. Verify that $H^2 \neq 0$ and $H^3 = 0$.

(6) Find a basis B of V such that $[H]_B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Exercise 4. (6-4-5-5)

Let V be an n-dimensional over a field $\mathbb{F}$ and T a linear operator on V.

(1) Prove by induction on $r \geq 2$ that if $c_1, c_2, \ldots, c_r$ are **distinct** characteristic values of T with associated characteristic vectors $v_1, v_2, \ldots, v_r$ (resp.), then $v_1, v_2, \ldots, v_r$ are linearly independent.

(2) Assume that $dim(range(T)) = r$. Prove that T has at most $r + 1$ characteristic values.

(3) Assume that $\mathbb{F} = \mathbb{R}$ and n is odd. Prove that there is **no linear operator** H on V with minimal polynomial $p_H(X) = X^2 + 2X + 17$.

(4) Assume that $\mathbb{F} = \mathbb{C}$. Construct a linear operator L on $V = \mathbb{C}^3$ with minimal polynomial $X^2 + 2X + 17$.