

King Fahd University of Petroleum and Minerals,  
Department of Mathematics- Term 221  
Exam 2 : Math 550, Linear Algebra  
Duration: 3 Hours

NAME :

ID :

Exercises 1:

Exercises 2:

Exercises 3:

Exercises 4:

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Total:

**Exercise 1.** (5-5-5-5)

Let  $V$  be a finite-dimensional vector space over a field  $F$  and  $T$  and  $H$  be two linear operators on  $V$ .

(1) Prove that if  $T$  and  $H$  are simultaneously diagonalizable, then  $T$  and  $H$  commute.

(2) Assume that  $T$  and  $H$  commute and  $H$  has exactly  $n$  distinct characteristic values. Prove that there is a basis  $B$  such that  $[T]_B$  and  $[H]_B$  are both diagonal.

Application: Set  $V = \mathbb{R}^3$  as a vector space over  $\mathbb{R}$ ,  $S$  its standard basis and let  $T(x, y, z) = (z, y, x)$  and  $H(x, y, z) = (y, x + z, y)$ .

(3) Find a basis  $B$  of  $V$  such that  $T$  and  $H$  are simultaneously diagonalizable.

(4) Find a matrix  $P$  such that  $P^{-1}[T]_S P$  and  $P^{-1}[H]_S P$  are both diagonal.

**Exercise 2.** (6-6-6-4-3 points)

Let  $V = \mathbb{R}^4$  and  $T$  be the linear operator on  $V$  defined by:

$T(x, y, z, t) = (2x + y, x + 2y, t, -z + 2t)$ . Use the standard basis to:

(1) Find the Smith normal form of  $xI - T$  and the invariant factors of  $T$ .

(2) Find the cyclic decomposition of  $\mathbb{R}^4$  under  $T$ .

(3) Find the primary decomposition of  $\mathbb{R}^4$  under  $T$ .

(4) Find the rational matrix form of  $T$ .

(5) Find the Jordan matrix form of  $T$ .

**Exercise 3.** (5-5-5-5-5)

Let  $V$  be an  $n$ -dimensional  $\mathbb{F}$ -vector space and  $T$  a linear operator on  $V$ .

(1) Assume that  $\mathbb{F} = \mathbb{C}$  and  $0$  is the unique characteristic value of  $T$ . Prove that  $T$  is Nilpotent.

(2) Assume that  $\mathbb{F} = \mathbb{R}$ . If  $0$  is the only characteristic value of  $T$ , is  $T$  Nilpotent? Justify.

(3) Assume that  $T$  has a cyclic vector. Prove that any linear operator  $H$  commuting with  $T$  is a polynomial in  $T$ .

Application: Set  $V = \mathbb{R}^3$  and let  $T$  be the linear operator given by  $T(x, y, z) = (x + y, z - y, 2z)$ .

(4) Find a cyclic vector of  $T$  (if it exists).

(5) Prove that for a linear operator  $U$  on  $V$ , every nonzero vector of  $V$  is a cyclic vector of  $U$  if and only if the characteristic polynomial of  $U$  is irreducible on  $F$ .

**Exercise 4.** (8-3-4 points)

Let  $V = \mathbb{R}^3$ .

- (1) Find the matrix, **in a rational form**, of a linear operator  $T$  on  $V$  with minimal polynomial  $P = (X - 1)(X - 2)$  and exactly Two invariant factors such that the real-valued function  $x \mapsto f(x)$ , where  $f(X)$  is the characteristic polynomial of  $T$ , has an inflection point at  $\frac{5}{2}$ .
- (2) Find the Jordan matrix form of  $T$ .
- (3) Prove that if every subspace of  $V$  is  $T$ -invariant, then  $T = cI$  for some  $c \in F$ .