King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 212

Final Exam : Math 550, Linear Algebra

Duration: 3 Hours

NAME :

ID :

Problem 1:

Problem 2:

Problem 3:

Problem 4:

Problem 5:

Problem 6:

Total:

Problem 1. (2-3-2-3-4)

Let V be an n-dimensional vector space over a field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), $S = \{e_1, \ldots, e_n\}$ a fixed basis of V and $\mathcal{M}_n(\mathbb{F})$ the $\mathbb{F}$-vector space of $n \times n$ matrices with coefficients in $\mathbb{F}$. For any vector $u \in V$, $[u]_S$ is the ($n \times 1$ matrix) coordinate of u in the basis S and $T_u : V \longrightarrow \mathcal{M}_n(\mathbb{F})$ defined by $T_u(v) = [u]_S[v]_S^t$ (t is the transpose operation) for every $v \in V$.

(1) Verify that T_u is a linear transformation.

(2) Prove that if $u \neq 0$, then T_u is one-to-one.

(3) Is T_u an isomorphism? Justify.

(4) Define $T : V \longrightarrow \mathcal{L}(V, \mathcal{M}_n(\mathbb{F}))$ (the $\mathbb{F}$-vector space of all linear transformations), by $T(u) = T_u$. Verify that T is a linear transformation.

(5) Find $rank(T)$ and $nullity(T)$.

Problem 2. (5-5 points)

Let V be an n-dimensional vector space over a field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), T a linear operator on V and W a T-admissible subspace of V.

(1) Prove that W has a T-invariant complement.

(2) Find an n-dimensional vector space V together with a linear operator T and a T-invariant subspace with a non-T-invariant complement.

Problem 3. (6-7 points)

Let V be an n-dimensional inner product space over a field $\mathbb{F}$ and let W be a subspace of V. Define a linear operator on V as follows: For each $x \in V$ uniquely expressible in the form $x = y + z$, where $y \in W$ and $z \in W^{\perp}$, set $U_W(x) = y - z$.

(1) Prove that U_W is both self-adjoint and unitary.

(2) Prove that every self-adjoint unitary linear operator on V is of this form (i.e. each linear operator that is self-adjoint and unitary on V is of the form U_W for some W to be determined).

Problem 4. (5-5-5-5 points)

Let V be a complex inner product space of finite-dimension and T a linear operator on V which admits an adjoint T^*.

(1) Assume that T is diagonalizable with characteristic values $\pm\sqrt[3]{2}$. Prove that $T^7 - 4T = 0$.

Assume that V is the inner product space of all $n \times n$ matrices with coefficients in $\mathbb{C}$, with the inner product $(A|B) = trace(AB^*)$, $\{B_{ij}\}_{1\leq i,j\leq n}$ its standard basis where B_{ij} takes 1 in the ith row and jth column and zero elsewhere, and D is a fixed diagonal matrix. Define a linear operator T_D on V by $T_D(A) = DA$ for every $A \in V$.

(2) Find the adjoint of T_D.

(3) Prove that T_D is a normal operator on V.

(4) Find the spectral resolution of T_D.

Problem 5. (5-5-4-4-5 points)

Let V be an n-dimensional complex inner product space, S a fixed basis of V, T a linear operator on V and $B = [T]_S$ (the matrix representing T in the basis S).

A/ Assume that T is non-singular.

(1) Prove that there is a lower triangular matrix M such that $MB = U$ is a unitary matrix.

(2) Application: Assume that $V = \mathbb{C}^3$, S its standard basis and set $T(x, y, z) = (\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y, y, z)$. Find M and U as in question (1).

B/ Assume that T is normal.

(3) Prove that characteristic vectors associated to distinct characteristic values are orthogonal.

(4) Prove that if T is Nilpotent, then $T = 0$ (the zero operator).

(5) Prove that if U is a normal operator commutating with T, then UT is normal.

Problem 6. (6-4-5-5)

Let V be an n-dimensional vector space over a subfield F of $\mathbb{C}$ and let f and g be skew-symmetric bilinear forms on V.

(1) Show that $rank(f) = rank(g)$ if and only if there is an invertible linear operator T on V such that $f(Tx, Ty) = g(x, y)$ for every $x, y \in V$.

Let h be the symmetric bilinear form on $V = \mathbb{R}^3$ given by:

$h(X, Y) = x_1x_2 + 2x_1y_2 + 2x_1z_2 + 2y_1x_2 + 4y_1y_2 + 8y_1z_2 + 2z_1x_2 + 8z_1y_2 + 4z_1z_2$ for every $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2)$.

(2) Find the quadratic form q associated to h and the matrix A representing h in the standard basis.

(3) Find an invertible matrix P such that $P^tAP = D$ is a diagonal matrix (not necessarily of characteristic values).

(4) Use the change of variable $X = PY$ to find the canonical form of q. What is the signature of q?