Math 550 - Exam II

150 minutes

Max Score: 50

Name:

ID:

Q1

(a) Define what we mean by a **diagonalizable** matrix, then give a **criterion** for the diagonalizability.

(b) Define what we mean by a **triangulable** matrix, then give a **criterion** for the triangulability.

(c) Give an example of a 2×2 matrix that is: **triangulable** but **not diagonalizable**.

Q2 State, without proof, the Primary Decomposition Theorem.

Q3 State, without proof, the Generalized Cayley-Hamilton Theorem.

Q4 Consider the matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(a) Find the Smith normal form of $xI - A$.

(b) Find the invariant factors of A.

(c) Find the rational form of A.

(d) Suppose that T is a linear operator such that $[T]_S = A$. Write an explicit cyclic decomposition of $\mathbb{R}^4$ under T.
Hint: The eigenvalues of A are:

1) $\lambda_1 = 0$, and $E_{\lambda_1} = Span \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

2) $\lambda_2 = 1$, and $E_{\lambda_2} = Span \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Q5 Consider the matrix:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 0 \end{bmatrix}, \quad a, b, c \in \mathbb{R}$$

1) Find all values of a, b, and c such that M is diagonalizable.

2) Suppose that $a = 1,\ b = 0$, and $c = 1$. That is, $M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

   Then: Find a diagonalizable matrix D and a nilpotent matrix N such that $M = D + N$ and $DN = ND$.

Q6 Find all possible Jordan forms of a matrix with characteristic polynomial $f = (x-2)(x-1)^2$. Then, distinguish the one(s) corresponding to the case when $nullity(T-I) = 1$.

BONUS QUESTION Consider the two matrices:

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

1) Prove that L and M are simultaneously diagonalizable.

2) Find a matrix P that diagonalizes both L and M. That is, find P such that: $P^{-1}LP$ is diagonal AND $P^{-1}MP$ is diagonal.

   Hint:
   The eigenvalues of L are: $\lambda_1 = 0$ with $am = 2$, and $\lambda_2 = 3$ with $am = 1$.
   The eigenvalues of M are: $\lambda_1 = 2$ with $am = 2$, and $\lambda_2 = 1$ with $am = 1$.