Math 550 - Final Exam

180 minutes

Max Score: 35

Name:

ID:

Q1. Let V be a vector space over $\mathbb{R}$. Let U_1, U_2, and U_3 be subspaces of V.

(a) Give an example to show that $U_1 \cap (U_2 + U_3) \neq (U_1 \cap U_2) + (U_1 \cap U_3)$.

(b) Prove that if $U_1 \subseteq U_3$, then $U_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap U_3$.

Q2.

(a) Define what an inner product is.

(b) Define what a normal operator is.

(c) Let V be a finite-dimensional inner product space and T a normal operator on V. Suppose $\alpha \in V$. Prove that α is an eigenvector for T associated with the eigenvalue c if and only if α is an eigenvector for T^* associated with the eigenvalue $\bar{c}$.

Q3. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. and define

$$T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$$

$$T(B) = AB - BA$$

(a) Find $G := [T]_S$, the matrix of T with respect to the standard basis of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(b) Find the minimal polynomial of G, knowing that the characteristic polynomial is $\det(\lambda I - G) = \lambda^2(\lambda - 4)(\lambda + 4)$.

(c) Is G diagonalizable? Justify.

(d) Find the Jordan form of G.

(e) Show that if $T(B) = nI$, then $n = 0$.

Q4. Suppose that $A_1, A_2, \cdots, A_8$ are eight 6×6 complex matrices,whose cube is zero (that is, $A_i^3 = 0$). Prove that two of these matrices must be similar.

Q5. Let V be a finite dimensional inner product space over $\mathbb{F}$, and T a self-adjoint linear operator on V.

(a) Prove that each eigenvalue of T is real.

(b) Prove that the eigenvectors of T associated with distinct eigenvalues are orthogonal.

Q6. Let V be a finite dimensional inner product space over $\mathbb{R}$,

(a) Give an example of a linear operator S on V such that $(Av|v) = 0, \forall v \in V$, but $(A^2v|v) \neq 0, \forall v \neq 0$.

(b) Prove that if S is a self-adjoint linear operator on V (hence S is symmetric), then $(Sv|v) = 0, \forall v \in V \implies S = 0$.

Q7. Consider the quadratic form $q \begin{bmatrix} x \\ y \end{bmatrix} = kx^2 - 4xy + ky^2$, $k \in \mathbb{R}$.

(a) Find k so that the quadratic form is positive definite.

(b) Suppose that $q = v^t D v$ where D is a diagonal matrix. Prove that the diagonal entries of D are all positive if and only if $k > 2$.

(c) Find k so that the symmetric bilinear form corresponding to q is non-degenerate.

(d) Let $k = 3$, find the symmetric bilinear form f corresponding to q.