King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 241

Exam 1 : Math 550, Linear Algebra

Duration: 3 Hours (+ 1H possible extension)

Full NAME :

ID :

Exercise 1 (out of 10)

Exercise 2 (out of 20)

Exercise 3 (out of 20)

Exercise 4 (out of 20)

Exercise 5 (out of 10)

Exercise 1. (3-3-4) [Section 2.3, Exercise 11, page 49]

Let $V = \mathcal{M}_2(\mathbb{C})$ be the **real-vector space** of all 2×2 matrices with complex entries, and let $W_1 = \{A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} | A_{11} + A_{22} = 0\}$,
$W_2 = \{A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} | A_{21} = -\overline{A_{12}}\}$, where the bar denotes the complex conjugate.

(1) Find a basis for W_1 and its dimension.

(2) Find a basis for W_2 and its dimension.

(3) Find a basis for $W_1 \cap W_2$ and its dimension.

Exercise 2. (3-4-4-3-6 points)

Let $V = \mathbb{R}^3$, T be the linear operator on V defined by $T(x, y, z) = (x, x + y, x + z)$.

(1) Find the matrix $[T]_S$ representing T in the standard basis S.

(2) Find the matrix $[T]_B$ representing T in the ordered basis $B = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$.

(3) Find explicitely the dual basis B^* of B.

(4) Let f, g, h be linear functionals in V^* defined by $f\,(x, y, z) = x + y + z$, $g\,(x, y, z) = x - 2y$, $h\,(x, y, z) = y - 3z$.

Verify that $\{f, g, h\}$ is a basis of the dual space V^*.

(5) Find the basis of V for which $\{f, g, h\}$ is the dual.

Exercise 3. (5-5-5-5 points)[Section 3.4, Exercise 9, page 96]

Let V be an n-dimensional vector space over a field $\mathbb{F}$ and T, H linear operators on V.

(1) Prove that if there are bases $\mathcal{B}$ and $\mathcal{B}'$ such that $[T]_{\mathcal{B}} = [H]_{\mathcal{B}'}$, then there is an invertible operator U such that $H = UTU^{-1}$.

(2) Conversely, prove that if $H = UTU^{-1}$ for some invertible operator U, then there are two bases $\mathcal{B}$ and $\mathcal{B}'$ such that $[T]_{\mathcal{B}} = [T]_{\mathcal{B}'}$.

Application: Assume that $V = \mathbb{R}^3$, and T and H are defined as follows: $T(x, y, z) = (y + z, x + z, x + y)$ and $H(x, y, z) = (x + 2y, x, x + y - z)$.

(3) Find (explicitely) two bases $\mathcal{B}$ and $\mathcal{B}'$ such that $[T]_{\mathcal{B}} = [H]_{\mathcal{B}'}$.

(4) Find (explicitely) an invertible operator U such that $H = UTU^{-1}$.

Exercise 4. (6-5-5-4) [Sections 3.2& 3.6, Exercises 12& 16, pages 84& 107]

Let p, m and n be positive integers, $V = \mathcal{M}_{m\times n}(\mathbb{F})$ be the vector space of $m \times n$ matrices and $W = \mathcal{M}_{p\times n}(\mathbb{F})$ be the vector space of $p \times n$ matrices over the field $\mathbb{F}$. Let B be a fixed $p \times m$ matrix and let $T : V \longrightarrow W$ be the linear operator defined by $T(A) = BA$.

(1) Prove that if T is invertible, then $p = m$ and B is an invertible $m \times m$ matrix.

(2) Prove that if $p = m$ and B is an invertible $m \times m$ matrix, then T is invertible and find its inverse.

(3) Assume that $p = n$ and let f be a linear functional on W such that $f(MN) = f(NM)$ for every matrices M and N in W. Prove that f is a scalar multiplication of the trace function. ($tr : W \longrightarrow \mathbb{F}$, $A \mapsto tr(A)$).

(4) Prove that if $f(I) = n$, then f is the trace function.

Exercise 5. (3-3-4)

Let V, W and U be three finite dimensional vector spaces over a field $\mathbb{F}$, $f : V \longrightarrow W$ and $g : W \longrightarrow U$ be linear transformations such that $gof = 0$.

(1) Prove that $rank(f) + rank(g) \leq dimW$.

(2) Prove that $dimV \leq Nullity(f) + Nullity(g)$.

(3) Assume that $V = W$ and $f^2 = 0$. Prove that $rank(f) \leq \frac{dimV}{2} \leq Nullity(f)$.