King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 241

Exam 2 : Math 550, Linear Algebra

Duration: 3 Hours

NAME :

ID :

Exercises 1 (out of 20):

Exercises 2 (out of 25):

Exercises 3 (out of 10):

Exercises 4 (out of 10):

Exercises 5 (out of 15):

Total:

Exercise 1. (5-5-5-5)

Let V be a finite-dimensional vector space over a field F and T and H be two linear operators on V.

(1) Prove that if T and H are simultaneously diagonalizable, then T and H commute.

(2) Assume that T and H commute and H has exactly n distinct characteristic values. Prove that there is a basis B such that $[T]_B$ and $[H]_B$ are both diagonal.

Application: Set $V = \mathbb{R}^3$ as a vector space over $\mathbb{R}$, S its standard basis and let $T(x, y, z) = (z, y, x)$ and $H(x, y, z) = (y, x + z, y)$.

(3) Find a basis B of V such that T and H are simultaneously diagonalizable.

(4) Find a matrix P such that $P^{-1}[T]_S P$ and $P^{-1}[H]_S P$ are both diagonal.

Exercise 2. (5-5-5-5-5 points)

Let $V = \mathbb{R}^4$ and T be the linear operator on V defined by:

$T(x, y, z, t) = (x, t, y + z, 0)$. Use the standard basis to:

(1) Find the Smith normal form of $xI - T$ and the invariant factors of T.

(2) Find the cyclic decomposition of $\mathbb{R}^4$ under T.

(3) Find the primary decomposition of $\mathbb{R}^4$ under T.

(4) Find projections E_i such that $V = \bigoplus range(E_i)$.

(5) Find the rational matrix form of T.

Exercise 3. (6-4 points)

Let $V = \mathbb{C}^6$, T a linear operator on V with minimal polynomial $(x-2)^2(x-3)$ and with **three** invariant factors.

(1) Find all possibilities of the matrix, **in a rational form**, representing T.

(2) Assume that the characteristic polynomial of T is $f(X) = (X - 2)^3(X - 3)^3$. Find the Jordan matrix form of T.

Exercise 4. (4-3-3)

Let V be an n-dimensional $\mathbb{F}$-vector space and T a linear operator on V.

(1) Assume that T is nilpotent and $T^{n-1} \neq 0$. Prove that T has a cyclic vector.

Application: Set $V = \mathbb{R}^3$ and let T defined by $T(x, y, z) = (y, z, 0)$.

(2) Verify that T is Nilpotent.

(3) Find a cyclic vector α of T and the matrix representing T in the basis of $Z(\alpha, T)$.

Exercise 5. (3-5-4-3) [Exercise 12, page 206 and Exercise 3, page 208]

Let V be an n-dimensional complex vector space, T a linear operator on V and $g(X)$

a nonzero polynomial.

(1) Prove that if c is a characteristic value of T, then $g(c)$ is a characteristic value of the operator $g(T)$.

(2) Prove that if λ is a characteristic value of $g(T)$, then $\lambda = g(c)$ for some characteristic value c of T.

(3) Assume that T has exactly n distinct characteristic values. Prove that every linear operator H on V commutating with T is a polynomial of T.

(4) Application: Assume that T has exactly two characteristic values 2 and 3. Let H be a linear operator commutating with T with nonzero characteristic values c_1 and c_2. Find explicitly a polynomial $h(X)$ of degree 1 such that $h(T) = H$.