King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 241

Final Exam : Math 550, Linear Algebra

Duration: 4 Hours

NAME :

ID :

Problem 1:———/18

Problem 2:———/15

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Total:——————/100

Problem 1. (4-3-4-4-3)

Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and V^* its dual space.

(1) Prove that $dimV = dimV^*$.

Set $V = \mathbb{R}^3$ and let $B = \{(1,0,1),(1,1,1),(0,0,1)\}$ and $B' = \{(1,0,1),(0,1,0),(-1,0,0)\}$ two bases for V.

(2) Find the transition matrix P from B to B'.

(3) Let T a linear operator on V with $[T]_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}$. Find $[T]_{B'}$.

(4) Find the dual basis B^* of B.

(5) Let $f : V \longrightarrow \mathbb{R}$ defined by $f(x,y,z) = x + y - z$. Express f is the basis B^*.

Problem 2. (4-3-4-4 points)

Let V be an inner product space over a field $\mathbb{F}$ and T be an invertible linear operator on V.

(1) Prove that there are (unique) unitary operator U and non-negative operator N such that $T = UN$.

Application: Let $V = \mathbb{R}^3$ be the real standard inner product space, $S = \{e_1, e_2, e_3\}$ its standard basis and and T the linear operator on V defined by: $T(x,y,z) = (4y, 4x, 5z)$.

(2) Verify that T is a diagonalizable normal operator.

(3) Find the spectral resolution of T.

(4) Find the Polar Decomposition of T.

Problem 3. (4-2-5-3-3-3 points)

Let V be an n-dimensional **complex inner product space**, T a linear operator on V, W a subspace of V and $W^\perp = \{y \in V | (x|y) = 0 \text{ for every } x \in W\}$. Define the function inner product $f : V \times V \longrightarrow \mathbb{C}$ by $f(x,y) = (x|y)$.

(1) Prove that f is a continuous functions. (Hint: use Cauchy-Schwartz inequality)

(2) Deduce that $W^\perp$ is closed (i.e. if $\{y_n\}_n$ is a sequence of elements of $W^\perp$ that converges to y, then $y \in W^\perp$.

(3) Prove that T is self adjoint if and only if $(T\alpha|\alpha)$ is a real number for every $\alpha \in V$.

(4) Prove that if T is self-adjoint, then the characterestic values of T are real numbers.

(5) Prove that if T is unitary, then $|c| = 1$ for every characterestic values of T.

(6) Find an example of a normal operator on a **real inner product space** that is not diagonalizable.

Problem 4. (5-4-4 points)

Let $V = \mathbb{P}_2 := \{f(X) \in \mathbb{R}[X] | deg(f) \leq 2\}$ be the real-inner product space of all polynomials of degree ≤ 2, with the scalar product $(f|g) = \int_0^1 f(x)g(x)dx$, and standard basis $S = \{1, x, x^2\}$. Let D be the differential operator on V and D^* its adjoint.

(1) Use Gram-Schimdt to find an orthonormal basis B of V.

(2) Find the matrices $[D]_B$ and $[D^*]_B$ representing D and D^* in the basis B.

(3) Is D a normal operator on V? Justify.

Problem 5. (3-3-4-4-4-4 points)

Let V be an n-dimensional **complex inner product space** ($n \geq 3$) and T a normal linear operator.

(1) Prove that $Nullspace(T) = Nullspace(T^*)$.

(2) Prove that $Nullspace(T)$ is the orthogonal complement of $range(T)$ (that is, $Nullspace(T) = (rang(T))^{\perp}$.

(3) Prove that for every polynomial $g(X)$, $g(T)$ is a normal operator.

(4) Suppose that there are two polynomials $f(X)$ and $g(X)$ relatively prime and $\alpha, \beta \in V$ such that $f(T)\alpha = g(T)\beta = 0$. Prove that $(\alpha|\beta) = 0$.

(5) Assme that T satisfies $T^n - 4T^{n-2} - T^2 + 4I = 0$. Prove that T is diagonalizable.

(6) Prove that there is a polynomial $h \in \mathbb{C}[X]$ such that $T^* = h(T)$.

Problem 6. (4-4-4)

Let V be a vector space over a field $\mathbb{F}$, f a bilinear forms on V and q its associated quadratic form.

(1) Suppost that there is $v \in V$ such that $q(v) \neq 0$. Let $W = span\{v\}$ and $W^{\perp} = \{u \in V | f(u, v) = 0\}$ its orthogonal. Prove that $V = W \oplus W^{\perp}$.

Assume that f is symmetric given by $f(X, Y) = x_1x_2 + 2x_1y_2 + 2x_1z_2 + 2y_1x_2 + 4y_1y_2 + 8y_1z_2 + 2z_1x_2 + 8z_1y_2 + 4z_1z_2$ for every $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2)$.

(2) Find the quadratic form q associated to f and the matrix A representing f in the standard basis.

(3) Find the canonical form of q. What is the signature of q?