King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 241

Exam 2 : Math 550, Linear Algebra

Duration: 3 Hours

NAME :

ID :

Problem 1 (out of 14):

Problem 2 (out of 20):

Problem 3 (out of 15):

Problem 4 (out of 15):

Problem 5 (out of 16):

Total:

Problem 1. (6-4-4)

Let $V = \mathbb{R}^3$, $S = \{e_1, e_2, e_3\}$ its standard basis, $\mathcal{L}(V,V)$ the space of all linear operators on V and W the subspace of $\mathcal{L}(V,V)$ defined by: $W = \{T \in \mathcal{L}(V,V) | TT' = T'T \text{ for all } T' \in \mathcal{L}(V,V)\}$.

(1) Prove that every $T \in W$ is of the form $T = cI$ for some constant $c \in \mathbb{R}$.

Set $T(x,y,z) = (x + y - z, 2y + z, 3z)$ and

$T'(x,y,z) = (4x + y - z, 5y + z, 6z)$.

(2) Show that T and T' are simultaneously diagonalizable.

(3) Find a basis B of V such that $[T]_B$ and $[T']_B$ are both diagonal.

Problem 2. (4-4-4-4-4 points)

Let $V = \mathbb{R}^4$ and T be the linear operator on V defined by:

$T(x,y,z,t) = (4z, x - 8z, y + 5z, t)$. Use the standard basis to:

(1) Find the Smith normal form of $xI - T$ and the invariant factors of T.

(2) Find the cyclic decomposition of $\mathbb{R}^4$ under T.

(3) Find the primary decomposition of $\mathbb{R}^4$ under T.

(4) Find projections E_i such that $V = \bigoplus range(E_i)$.

(5) Find the rational matrix form of T.

Problem 3. (3-6-6 points)

Let $V = \mathbb{C}^6$, T a linear operator on V with minimal polynomial $(x+2)^2(x-1)^2$ and with **three** invariant factors.

(1) Find all possible invariant factors.

(2) Find all possibilities of the matrix, **in a rational form**, representing T.

(3) Find all possible Jordan matrix forms of T.

Problem 4. (3-5-4-3) [Exercise 12, page 206 and Exercise 7, page 231]

Let V be an n-dimensional complex vector space, T a linear operator on V and $g(X)$ a nonzero polynomial.

(1) Prove that if c is a characteristic value of T, then $g(c)$ is a characteristic value of the operator $g(T)$.

(2) Prove that if λ is a characteristic value of $g(T)$, then $\lambda = g(c)$ for some characteristic value c of T.

Now Assume that T is diagonalizable.

(3) Prove that if T has a cyclic vector, then T has exactly n distinct characteristic

values.

(4) Assume that T has exactly n characteristic values abd $B = \{u_1, \dots, u_n\}$ a basis of characterestic vectors. Prove that $\alpha = u_1 + \cdots + u_n$ is a cyclic vector for T.

Problem 5. (4-4-4-4)

Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and T a linear operator on V. Let p_o denote the minimal polynomial of T and consider the primary decomposition for T given by

$$\begin{cases} V = W_1 \oplus \cdots \oplus W_k \\ p_o \;=\; p_1^{r_1} p_2^{r_2} \dots p_k^{r_k} \;, \text{ where } p_1, p_2, \dots, p_k \text{ are distinct } \textit{irreducible} \text{ monic polynomials} \\ W_i := Nullspace(p_i^{r_i}(T), \;\forall \text{(i)} \\ MinPoly(T_i) = p_i^{r_i} \;, \;\forall\, i \;\text{ where } T_i \text{ denotes the restriction of } T \text{ on } W_i \end{cases}$$

(1) Prove that if W is an *invariant* subspace, then $W = (W \cap W_1) \oplus \cdots \oplus (W \cap W_k)$. [Hint: Use projections]

(2) Prove that if p_o is *irreducible*, then every *invariant* subspace is T-admissible.

(3) Prove that every *invariant* subspace is T-admissible $\iff p_o = p_1 p_2 \cdots p_k$ (i.e., $r_i = 1 \;, \;\forall\, i$).

(4) Suppose $\mathbb{F} = \mathbb{C}$. Deduce from (3) that: T is diagonalizable $\iff$ Every *invariant* subspace is T-admissible.