King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 251

Final Exam : Math 550, Linear Algebra

Duration: 4 Hours

NAME :

ID :

Problem 1:———/12

Problem 2:———/12

Problem 3:———/20

Problem 4:———/16

Problem 5:———/20

Problem 6:———/20

Total:————————/100

Problem 1. (4-4-4)

Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and V^{**} its **double dual** space.

(1) Find an isomorphism φ from V to V^{**}.

Set $V = \mathbb{R}^3$ and let B the oredered basis given by $B = \{(1,2,1),(2,1,2),(1,0,0)\}$

(4) Find the dual basis B^* of B.

(5) Let $f : V \longrightarrow \mathbb{R}$ defined by $f(x,y,z) = x+y+z$. Express f is the basis B^*.

Problem 2. (4-4-4 points)

Let $V = \mathbb{R}^3$ be the real standard inner product space, $S = \{e_1, e_2, e_3\}$ its standard basis and and T the linear operator on V defined by: $T(x,y,z) = (2x, 5z, 5y)$.

(1) Verify that T is a diagonalizable normal operator.

(2) Find the spectral resolution of T.

(3) Find the Polar Decomposition of T.

Problem 3. (4-3-5-4-4 points)[Exercise 7, page 347 and Exercise 9 page 348]

Let $V = \mathcal{M}_n(\mathbb{C})$ be the space of complex matrices equiped with the inner product $(A|B) = trace(AB^*)$, D a diagonal matrix, P a unitary matrix in V, T_D and H_P the linear operators on V defined by $T_D(A) = DA - AD$; and $H_P(A) = PAP^{-1}$.

(1) Find the adjoint T_D^* of T_D.

(2) Show that T_D is self-adjoint if and only if $D = D^*$.

(3) Assume that $D = D^*$. Find the spectral decomposition of T_D.

(4) Prove that H_P is a unitary operator on V.

(5) Set $n = 2$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Find explicitely the spectral decomposition of T_D.

Problem 4. (4-4-4-4 points)

Let $V = \mathbb{P}_2 := \{f(X) \in \mathbb{R}[X] | deg(f) \leq 2\}$ be the real-inner product space of all polynomials of degree ≤ 2, with the scalar product $(f|g) = \int_0^1 f(x)g(x)dx$, and standard basis $S = \{1, X, X^2\}$. Let W be subspace of V spaned by $\{1, 2X-1\}$ and T the linear operator on V defined by $T(a_0 + a_1X + a_2X^2) = a_0 + a_1X$ and T^* its adjoint.

(1) Use Gram-Schimdt to find an orthonormal basis B of V.

(2) Find the matrices $[T]_B$ and $[T^*]_B$ representing T and T^* in the basis B.

(3) Is T a normal operator on V? Justify.

(4) Find the best approximation of $f = X^2$ on W.

Problem 5. (4-4-4-4-4 points)

Let V be an n-dimensional **complex inner product space** ($n \geq 3$), T a linear operator on V and T^* its adjoint.

(1) Prove that c is a characteristic value of T if and only if $\overline{c}$ is a characteristic value of T^*.

(2) Prove that if $\{v_1 \ldots, v_s\}$ is a family of characteristic vectors associated to distinct characteristic values $c_1, \ldots, c_s$ respectively, then $\{v_1, \ldots, v_s\}$ are linearly independent.

(3) Assume that $(Tx|x) \geq 0$ for every $x \in V$. Prove that every characteristic value of T is positive.

(4) Assume that $T = T^*$ and all characteristic values are positive. Prove that $(Tx|x) \geq 0$ for every $x \in V$.

(5) Prove that for every positive operator H, $tr(TH) \geq 0$.

Problem 6. (4-5-3-4-4)[Exercises 9 and 10 page 378]

Let V be a finite-dimensional vector space over $\mathbb{F} \subseteq \mathbb{C}$ and let L_1 and L_2 be two *nonzero* linear functionals on V. Consider the skew symmetric bilinear form

$$f(\alpha, \beta) = L_1\alpha\ L_2\beta\ -\ L_1\beta\ L_2\alpha\ ,\ \forall\ \alpha, \beta \in V$$

(1) Show that L_1 and L_2 are linearly dependent $\Longleftrightarrow\ f = 0$.

(2) Let g be a skew symmetric bilinear form on V. Show that $rank(g) = 2$ if and only if $g(\alpha, \beta) = L_1\alpha\ L_2\beta\ -\ L_1\beta\ L_2\alpha$ for some linear functionals L_1 and L_2 on V.

Next, set $V = \mathbb{F}^3$, $L_1(x, y, z) = x + y + z$ and $L_2(x, y, z) = x - y + z$.

(3) Find the matrix of f in the standard ordered basis $S := \{e_1, e_2, e_3\}$ and find the rank of f.

(4) Consider the bilinear form h on V defined by

$$h(\alpha, \beta) = x_1y_1 + x_1y_2 - 3x_1y_3 + x_2y_1 + x_2y_2 - 3x_2y_3 - 3x_3y_1 - 3x_3y_2 + 5x_3y_3$$

for every $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3)$.

Find the matrix A representing h in the standard basis, and the quadratic form q associated to h.

(5) Find the canonical form of q. What is the signature of q?