KFUPM/ Department of Mathematics/T231/MATH 551/ Exam 1/

Name: **ID #:**

[*Justify all answers.*]

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1. [12pts] (a) Let R be a ring with $1 \neq 0$. Show that if R is a division ring, then R has exactly two left ideals. Is the converse true?

(b) Let $f : R \longrightarrow S$ be a ring homomorphism, where R and S are commutative rings. Prove that if Q is a prime ideal of S and $f^{-1}(Q) \neq R$, then $f^{-1}(Q)$ is a prime ideal of R containing $\ker f$.

(c) Is there a ring homomorphism $g : \mathbb{C} \longrightarrow \mathbb{R}$ such that $f(1) = 1$?

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2. [8pts] Let R be a ring with $1 \neq 0$ and let A be a submodule of a left R-module B.

(a) Prove that if B is finitely generated, then so too is B/A.

(b) Is the subset I of R defined as $\{r \in R : rB \subseteq A\}$ a right ideal of R? a left ideal of R?

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3. [8pts] (a) Let $f : A \longrightarrow B$ be a homomorphism of left R-modules and let C be a submodule of A. Prove that $f^{-1}(f(C)) = C + \ker f$.

(b) Let M be a left R-module with submodules A and B and let $f : M \longrightarrow N$ be an R-module homomorphism. Suppose $A \cap B = \ker f$.

(i) Prove that $f(A) \cap f(B) = 0$.

(ii) Is it true that $f(A + B) = f(A) + f(B)$ (d.s.) ?

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4. [8pts] (a) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of left R-modules and R-homomorphisms, where f is surjective and g is injective. Prove that $B = 0$.

(b) Let A and B be submodules of a left R-module M. Consider the sequence

$$0 \longrightarrow A \cap B \xrightarrow{f} A \times B \xrightarrow{g} A + B \longrightarrow 0$$

where f and g are the R-homomorphisms given $f(x) = (x, x)$ and $g(x, y) = x - y$. Prove that this sequence is exact.