**King Fahd University of Petroleum & Minerals**

**Department of Mathematics and Statistics**

**Math 551: Abstract Algebra**

**First Exam, Spring Semester 232 (120 minutes)**

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**Remark:** Solve **6** questions including **Q7**. Show **full details.**

**Q1. (16 points)**

(a) Prove that a ring $R$ is left Noetherian if and only if every left ideal of $R$ is finitely generated.

(b) Show that the following ring is **not** left Noetherian

$$R := \left\{ \begin{bmatrix} m & p \\ 0 & q \end{bmatrix} \mid m \in \mathbb{Z},\ p, q \in \mathbb{Q} \right\}.$$

(c) Give an example of a ring that is **not** right Noetherian.

**Q2. (16 points)** Characterize the integers $n$ for which the ideals $(n)$ of $\mathbb{Z}$ are:

(a) prime

(b) maximal

(c) semiprime

(d) primary

**Q3. (16 points)** Let $R$ be a commutative ring.

(a) Show that a proper ideal $I$ of $R$ is primary if and only if every every zerodivisor in $R/I$ is nilpotent.

(b) If $f : R \longrightarrow S$ is a morphism of commutative rings and $Q$ is a $P$-primary ideal of $S$, then $f^{-1}(Q)$ is an $f^{-1}(P)$-primary ideal of $R$.

**Q4. (16 points)** Show that:

(a) If $\mathfrak{m}$ is a maximal ideal of a commutative ring $R$, then $\mathfrak{m}^n$ is $\mathfrak{m}$-primary for all integers $n \geq 1$.

(b) Show that the ideal $P :=< [x], [y] >$ of $\mathbb{R}[x, y, z]/(xz - y^2)$ is prime but not primary.

**Q5. (16 points) (a)** Show that the category **Ring** of rings (with identity) is complete.

(b) Prove that a category $\mathcal{C}$ with *binary* coproducts and pushouts have coequalizers.

**Q6. (16 points)** Consider for each vector space $V$ over a field $\mathbb{K}$ its *dual double* $V^{**} := Lin_{\mathbb{K}}(Lin_{\mathbb{K}}(V, \mathbb{K}), \mathbb{K})$ and let

$$\mathcal{F} := (-)^{**} : \mathbf{Vec}(\mathbb{K}) \longrightarrow \mathbf{Vec}(\mathbb{K}).$$

(a) Show that $\mathcal{F}$ is a covariant functor.

(b) Show that the restriction

$$F : \mathbf{FDVec}(\mathbb{K}) \longrightarrow \mathbf{FDVec}(\mathbb{K})$$

of $(-)^{**}$ to the subcategory of finite dimensional real vector spaces $\mathbf{FDVec}(\mathbb{K}) \subset \mathbf{Vec}(\mathbb{K})$ is an *equivalence* of categories.

**Q7. (20 points)** Prove or disprove:

(a) Every semiprime primary ideal of a commutative ring is prime.

(b) Every monomorphism in Grps is a *normal monomorphism* (i.e. a kernel of a pair of group homomorphisms).

(c) $\{\mathbb{R}^{(\alpha)} \mid \alpha \text{ is a cardinal number}\}$ is a skeleton of $\mathbf{Vec}(\mathbb{R})$.

(d) Every epimorphism of groups has a right inverse.

# GOOD LUCK