King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 551: Abstract Algebra

First Exam, Spring Semester 232 (120 minutes)

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Remark: Solve **6** questions including **Q7**. Show **full details.**

Q1. (16 points) Let $\{M_i\}_{i\in I}$ be a collection of left R-modules. Show that

(a) If $\{A_i\}_{i\in I}$ is such that $A_i \leq_R M_i$ is an R-submodule for each $i \in I$, then we have isomorphisms of left R-modules:

$$\left(\prod_{i\in I} M_i\right) / \left(\prod_{i\in I} A_i\right) \simeq \prod_{i\in I} M_i/A_i.$$
$$\left(\bigoplus_{i\in I} M_i\right) / \left(\bigoplus_{i\in I} A_i\right) \simeq \bigoplus_{i\in I} M_i/A_i;$$

(b) For very left R-module X, we have isomorphisms of Abelian groups

$$Hom_R(X, \prod_{i\in I} M_i) \simeq \prod_{i\in I} Hom_R(X, M_i);$$
$$Hom_R(\bigoplus_{i\in I} M_i, X) \simeq \prod_{i\in I} Hom_R(M_i, X).$$

Q2. (16 points)

(a) Show that every left R-module M has a maximal linearly independent subset, and that a maximal linearly independent subset of M generates an **essential** submodule $S \trianglelefteq_R M$ (i.e. $S \cap A \neq 0$ for every $0 \neq A \leq_R M$).

(b) Every subspace W of a vector space V we have

$$dim(W) + dim(V/W) = dim(V).$$

(c) Every (finitely generated) left R-module M over a ring R is the homomorphic image of a (finitely generated) free left R-module F.

(d) R is a division ring if and only if every left (right) R-module is free.

Q3. (16 points) Show that for a left R-module M :

(a) $Soc(Soc({}_RM)) = Soc({}_RM)$.

(b) If $f : M \longrightarrow N$ is R-linear, then $f(Soc({}_RM)) \subseteq Soc({}_RN)$.

(c) If $K \leq_R M$, then $Soc({}_RK) = K \cap Soc({}_RM)$.

(d) If ${}_RM$ is Artinian, then $Soc({}_RM) \trianglelefteq M$ (an **essential** R-submodule).

Q4. (16 points)
(a) Show that for any collection $\{M_i\}_{i\in I}$ of left R-modules:

$$Soc(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} Soc(M_i).$$

(b) Show that for every positive integer n, there exists a *unique* semisimple Abelian group of order n.
(c) Find the unique semisimple Abelian group of order 180.
(d) Find $Soc(\mathbb{Z}_{180})$.

Q5. (16 points) Let R be a left Artinian ring. Show that:

(a) $Jac(R)$ is nilpotent.
(b) $R/Jac(R)$ is semisimple.
(c) R is left Noetherian.
(d) If R has no non-zero right zerodivisors, then R is a division ring. [**Hint:** For $a \in R\backslash\{0\}$, consider the R-linear map $\rho_a : {}_RR \longrightarrow {}_RR$, $r \longmapsto ra$].

Q6. (16 points) Show that:

(a) Show that a semisimple module is of finite length if and only if it is finitely generated.
(b) If ${}_RM$ is a left R-module and $K \leq_R M$, then ${}_RM$ has finite length if and only if K and M/K have finite length; and in this case

$$lg(M) = lg(K) + lg(M/K).$$

(c) Provide four composition series of $\mathbb{Z}_{24}$ as a $\mathbb{Z}$-module and illustrate the equation above for $\mathbb{Z}$-submodule $K = \{0, 6, 12, 18\}$ of $M = \mathbb{Z}_{24}$.
(d) Show that $Soc({}_{\mathbb{Z}}\mathbb{Z}_4) = Rad({}_{\mathbb{Z}}\mathbb{Z}_4)$.

Q7. (20 points) Prove or disprove (R is an associative ring):
(a) Every simple ring is a division ring.
(b) For every finitely generated non-zero left R-module M and every left ideal $I \subseteq Jac(R)$, we have $IM \neq M$.
(c) $Jac(R)Soc(M) = 0$ for every left R-module M.
(d) Every Artinian left R-module is Noetherian.

GOOD LUCK

Handout

Definition: Define an R-submodule $L \leq M$ to be

- **superfluous (small),** and we write $L << M$, iff for every $K \lneqq_R M$: the equality $K + L = M$ implies $K = M$.

- **essential (large)**, and we write $L \trianglelefteq M$, iff for every $0 \neq K \leq_R M$, we have $K \cap L \neq 0$.

Moreover, set

$$\mathcal{L}_M := \{L \leq_R M \mid \ _RL \text{ is essential}\}; \qquad \mathcal{S}_M := \{L \leq_R M \mid \ _RL \text{ is simple}\};$$
$$\mathfrak{S}_M := \{L \leq_R M \mid \ _RL \text{ is superfluous}\}; \qquad \mathcal{M}_M := \{L \leq_R M \mid \ _RL \text{ is maximal}\}.$$

and

$$\begin{aligned} Soc(_RM) &= \bigcap_{L \in \mathcal{L}_M} L = \sum_{L \in \mathcal{S}_M} L = \ (:= 0 \text{ if } \mathcal{S}_M = \varnothing); \\ Rad(_RM) &= \sum_{L \in \mathfrak{S}_M} L \ = \bigcap_{L \in \mathcal{M}_M} L = (:= M \text{ if } \mathcal{M}_M = \varnothing). \end{aligned}$$