King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 551: Abstract Algebra

Final Exam, Spring Semester 232 (180 minutes)

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Remark: Solve **7** questions including **Q8 & Q9**. Show **full details.**

Throughout, R is an associative ring with 1.

Q1. (16 points) Show that

(a) A left R-module P is projective if and only P is a direct summand of a free left R-module F.

(b) The direct sum $\bigoplus_{\lambda\in\Lambda} P_\lambda$ of projective left R-modules $\{P_\lambda\}_{\lambda\in\Lambda}$ is projective.

(c) A left R-module P is projective if and only if the functor

$$Hom_R(P,-): \ R\text{-}\mathbf{Mod} \longrightarrow \mathbb{Z}\text{-}\mathbf{Mod}$$

is exact.

(d) Every projective left R-module P is flat.

Q2. (16 points) Show that

(a) If a left R-module J is a direct summand of an injective left R-module K, then J is injective.

(b) The direct product $\prod_{\lambda\in\Lambda} J_\lambda$ of injective left R-modules $\{J_\lambda\}_{\lambda\in\Lambda}$ is injective.

(c) A left R-module J is injective if and only if the functor

$$Hom_R(-,J): \ R\text{-}\mathbf{Mod} \longrightarrow \mathbb{Z}\text{-}\mathbf{Mod}$$

is exact.

(d) Every injective left R-module J is divisible (i.e. for every $r \in R$ which is not a zero-divisor and every $b \in J$, there exists some $a \in J$ such that $ra = b$).

Q3. (16 points) Show that

(a) If a left R-module E is a direct summand of a flat left R-module F, then E is flat.

(b) The direct sum $\bigoplus_{\lambda \in \Lambda} F_\lambda$ of flat left R-modules $\{F_\lambda\}_{\lambda\in\Lambda}$ is flat.

(c) A left R-module F is flat if and only if the functor

$$- \otimes_R F : \ \textbf{Mod-}R \longrightarrow \mathbb{Z}\textbf{-Mod}$$

is exact.

(d) If a left R-module F is flat and D is a divisible Abelian group, then the right R-module $Hom_{\mathbb{Z}}(F, D)$ is injective.

Q4. (16 points) Show that

(a) $R/I \otimes_R M \simeq M/IM$ for any ideal I of R and any left R-module M.

(b) $R/I \otimes_R R/J \simeq R/(I + J)$ for any ideals I, J of R.

(c) $A \otimes_R B$ a projective Abelian group for any projective right R-module A and projective left R-module B.

(d) If A is a right T-module and B is a left T-module, then every morphism of rings $\rho : R \longrightarrow T$ induces a natural epimorphism of Abelian groups

$$A \otimes_R B \longrightarrow A \otimes_T B$$

Q5. (16 points) Let R be commutative, $S \subseteq R$ be a proper multiplicative subset and M, N be R-modules. Show that:

(a) $S^{-1}R \otimes_R M \simeq S^{-1}M$ for every R-module M.

(b) $S^{-1}(M \otimes_R N) \simeq S^{-1}M \otimes_{S^{-1}R} S^{-1}N$.

(c) $S^{-1}R$ is flat as an R-module.

(d) If F is a flat R-module, then $S^{-1}F$ is flat as an $S^{-1}R$-module.

Q6. (16 points) Compute (showing all details)

(a) $\mathbb{Z}_8 \otimes_{\mathbb{Z}} \mathbb{Z}_{20}$

(b) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$

(c) $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$

(d) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$

Q7. (16 points) Consider the commutative diagram of left R-modules and R-linear maps with exact rows and columns

$$
\begin{array}{ccccc}
Ke\,\varphi_1 & & Ke\,\varphi_2 & & Ke\,\varphi_3 \\
\downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 \\
M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 \\
\downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 \\
N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \\
\downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 \\
Coke\,\varphi_1 & & Coke\,\varphi_2 & & Coke\,\varphi_3
\end{array}
$$

Show that

(a) There exist uniquely determined R-linear maps

$$
\begin{gathered}
Ker(\varphi_1) \xrightarrow{\alpha_1} Ker(\varphi_2) \xrightarrow{\alpha_2} Ker(\varphi_3) \\
CoKer(\varphi_1) \xrightarrow{\beta_1} CoKer(\varphi_2) \xrightarrow{\beta_2} CoKer(\varphi_3)
\end{gathered}
$$

(b) If g_1 is a monomorphism, then the first row is exact.

(c) If f_2 is an epimorphism, then the last row is exact.

(d) If g_1 is a monomorphism and f_2 is an epimorphism, then there exists a (connecting) R-linear map $\delta : Ker(\varphi_3) \longrightarrow CoKer(\varphi_1)$ which yields an exact sequence of left R-modules

$$
Ker(\varphi_2) \xrightarrow{\alpha_2} Ker(\varphi_3) \xrightarrow{\delta} CoKer(\varphi_1) \xrightarrow{\beta_1} CoKer(\varphi_2).
$$

Q8. (12 points) Prove or disprove:

(a) Every submodule of a finitely generated left R-module is finitely generated.

(b) For any ideal I of R, we have $Jac(R/I) = (Jac(R) + I)/I$.

(c) $\mathbb{Z}_4$ is projective as a $\mathbb{Z}_{12}$-module.

(d) Every submodule of a free left R-module is free.

Q9. (8 points) TRUE or FALSE:

1. If M is a free left R-module, then all bases of M have the same cardinality.
2. A ring R is semisimple if and only if $R \simeq R_1 \times \cdots \times R_1$ a finite direct product of simple rings.
3. Every left R-module M can be embedded as an essential submodule of an injective left R-module $E(M)$.
4. If R is left Artinian, then the direct product $\prod\limits_{\lambda \in \Lambda} P_\lambda$ of projective left R-modules $\{P_\lambda\}_{\lambda \in \Lambda}$ is projective.
5. A ring R is left semisimple if and only if R is right Artinian and Jacobson-seimisimple.
6. $M \simeq M^{**}$ for every finitely generated left R-module M.
7. For any left R-module M, the functor
$$- \otimes_R M : \ \mathbf{Mod}\text{-}R \longrightarrow \mathbb{Z}\text{-}\mathbf{Mod}$$
preserves equalizers.
8. Every Artinian commutative domain is a field.

GOOD LUCK