King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 555: Commutative Algebra

Final Exam, Fall Semester 251 (180 minutes)

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Remark: Solve **8** questions including **Q8, Q9** & **Q10**. Show **full details.** All rings are commutative and non-zero.

Q1. (10 points) Let A be a subring of the ring B. Show that

1. If C is the *integral closure* of A in B and $S \subseteq A$ is a multiplicatively closed subset, then $S^{-1}C$ is the *integral closure* of $S^{-1}A$ in $S^{-1}B$.
2. If B is integral over A, then the *Jacobson radical* of A is the contraction of the *Jacobson radical* of B.

Q2. (10 points) Let A be a *Jacobson ring* (*i.e.*, every prime ideal P of A is the intersection of the *maximal* ideals of A containing P). Show that

1. A/I is a Jacobson ring for every ideal I of A.
2. If B is an A-algebra that is integral over A, then B is a Jacobson ring.

Q3. (10 points) Let A be a Noetherian ring. Show that

1. $Spec(A)$ is a Noetherian topological space (*i.e.*, the open subsets of $Spec(A)$ satisfy the ascending chain condition).
2. A is an *FMin-ring* (*i.e.*, A has only *finitely many minimal prime* ideals).

Q4. (10 points) Let A be a Noetherian ring. Show that

1. A is *coherent* (*i.e.*, every finitely generated ideal of A is finitely presented).
2. Every irreducible ideal of A is primary.

Q5. (10 points) Let A be an Artinian ring. Show that

1. $Jac(A)$ is nilpotent.
2. A is an *FMax-ring* (*i.e.*, A has only *finitely many maximal* ideals).

Q6. (10 points) Let A be a Dedekind domain (*i.e.*, A is an integrally closed Noetherian integral domain with $K.\dim(A) = 1$). Show that

1. If $S \subseteq A$ is a multiplicatively closed subset, then $S^{-1}A$ is either a Dedekind domain or the field of fractions of A.
2. The lattice $(Ideal(A), +, \cap)$ is *distributive*, *i.e.*, for any ideals I, J and K of A, we have

$$I \cap (J + K) = (I \cap J) + (I \cap K).$$

Q7. (10 points) Let V be a valuation ring (*i.e.*, an integral domain with field of fractions $\mathbb{K}$ such that for every $x \in \mathbb{K} \backslash \{0\}$, either $x \in V$ or $x^{-1} \in V$, or both). Show that

1. V is a *uniserial* ring (i.e. the ideals of V are totally ordered).
2. If V is Noetherian, then V is a DVR.

Q8. (15 points) Find (showing full details):

1. The *isolated* associated prime ideal(s) of $< 2x^2 + 3x + 1, x^2 + 3x + 2 >$ in $\mathbb{Z}[x]$.
2. The (6)-adic completion of $\mathbb{Z}$.
3. The first 3 terms of $\sqrt[3]{2} = (a_n)_{n \geq 0}$ in the ring of 5-*adic integers.*

Q9. (15 points) Prove *or* disprove:

1. Any *subring* of a Noetherian ring is Noetherian.
2. If M is an A-module and N_1, N_2 are A-submodules of M such that M/N_1 and M/N_2 are Artinian, then $M/(N_1 \cap N_2)$ is Artinian.
3. Any product of Artinian local rings is Artinian.

Q10. (20 Points) For each of the statements below, indicate whether it is **TRUE** or **FALSE**:

1. If A be a subring of a ring B such that B is integral over A, $\mathfrak{n}$ a maximal ideal of B and $\mathfrak{m} := A \cap \mathfrak{n}$, then $B_{\mathfrak{n}}$ is integral over $A_{\mathfrak{m}}$.
2. If M is an Artinian A-module, then $A/ann_A(M)$ is an Artinian ring.
3. For any prime number p, the ring of p-adic integers is a DVR.
4. There is *no* ring A such that $A \simeq A \times A$ as rings.
5. If V is a valuation ring and $\mathfrak{p}$ is a prime ideal of V, then $V_{\mathfrak{p}}$ is a valuation ring of its field of fractions.
6. If A is a Noetherian ring, then the ring $A[[x]]$ of formal power series is Noetherian.
7. $\mathbb{Z}[\sqrt{-3}]$ is a Dedekind domain.
8. If $A[x]$ is Noetherian, then A is Noetherian.
9. Noetherianity of rings is a local property.
10. Every Artinian module is Noetherian.

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