King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 555: Commutative Algebra

Midterm Exam, Fall Semester 251 (150 minutes)

Prof. Jawad Abuihlail

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Remark: Solve **6** questions including **Q6** & **Q7**. Show **full details.** All rings are commutative and non-zero.

Q1. (15 points) Let R be a commutative ring and consider the prime spectrum $X = Spec(R)$ with the Zariski topology. Show that the maps

$$I \longmapsto V(I) \text{ and } Y \longmapsto \bigcap_{P \in Y} P$$

provide

(a) $1-1$ correspondence between $\{I \leq_R R \mid I = \sqrt{I}\}$ and the collection of *closed sets* in X.

(b) $1-1$ correspondence between $Spec(R)$ and the collection of *irreducible closed sets* in X.

(c) $1-1$ correspondence between $Min(R)$ and the collection of *irreducible components* of X.

Q2. (15 points) Let M be a module over a commutative ring R. Show that:

(a) There exists a surjective R-linear map $\psi : R^m \longrightarrow M$ with $Ker(\psi)$ finitely generated as an R-module if and only if M is *finitely presented* (i.e. there exists an exact sequence of R-modules $R^n \longrightarrow R^m \longrightarrow M \longrightarrow 0$).

(b) If $_RM$ is finitely presented and $S \subseteq R \backslash \{0\}$ is a multiplicatively closed subset, then

$$S^{-1}Hom_R(M, N) \simeq Hom_{S^{-1}R}(S^{-1}M, S^{-1}N).$$

(c) If $_RM$ is finitely generated and $\phi : M \longrightarrow R^k$ is surjective, then $Ker(\phi) \leq_{\oplus} M$ is a direct summand and finitely generated.

Q3. (15 points) Show that the following are equivalent for a commutative ring R :

(a) R is absolutely flat (i.e. every R-module is flat).

(b) Every principal ideal of R is idempotent.

(c) Every finitely generated ideal of R is a direct summand.

Q4. (15 points) Let R be an integral domain with field of fractions K.
(a) Show that $R_P \hookrightarrow K$ canonically for every $P \in Spec(R)$.
(b) Show that $R = \bigcap_{P \in Spec(R)} R_P = \bigcap_{\mathfrak{m} \in Max(R)} R_{\mathfrak{m}}$.
(c) Find $Tor^1_R(K, R \oplus K)$.

Q5. (15 points) Let R be an integral domain and $S \subseteq R \backslash \{0\}$ be a multiplicatively closed subset. Call $I \leq_R R$ an S-*large ideal* iff every $\mathfrak{m} \in V(I) \cap Max(R)$ is disjoint from S. Show that

(a) If I is an S-large ideal of R, then $S^{-1}I \cap R = I$.

(b) There is morphism of *monoids* between the set of S-large ideals of R and the set of ideal of $S^{-1}I$ (where the operations are multiplication of ideals).

(c) If I is an S-large ideal of R, then there is a *natural* isomorphism of rings

$$R/I \simeq S^{-1}R/S^{-1}I.$$

Q6. (16 points) Compute *up to isomorphism* (showing full details):
(a) $\mathbb{Z}_{12} \otimes_{\mathbb{Z}} \mathbb{Z}_{30}$
(b) $S^{-1}R[x]$, where $S = \{x^n | \ n \geq 0\}$
(c) $(\mathbb{Q}[x]_{(x-3)}, \mathfrak{m})$ and $\mathbb{Q}[x]_{(x-3)}/\mathfrak{m}$ (where $\mathfrak{m}$ is the maximal ideal of $\mathbb{Q}[x]_{(x-3)}$)
(d) $Jac(\mathbb{R}[[x]])$

Q7. (24 points) Prove or disprove:

(a) Being an integral domain is a local property.

(b) For any proper ideal I of a commutative ring R, the variety $V(I)$ has minimal elements (with respect to inclusion).

(c) If R is a commutative ring, M is an R-module, I is a *finitely generated* ideal of R such that $I \subseteq Jac(R)$ and $IM = M$, then $M = 0$.

(d) If R is a commutative ring and $S \subseteq R \backslash \{0\}$ is a multiplicatively closed subset, then the ring $S^{-1}R$ is local.

GOOD LUCK