King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 555 (Term 222)

Mid-term Exam (Duration = 6 hours)

Problem 1. [10]

Let A be a commutative ring and Spec(A) the spectrum of A endowed with Zariski topology. A topological space Y is *irreducible* if Y is not empty and any two non-empty open subsets of Y intersect or, equivalently, if $Y = V_1 \cup V_2$ with V_1 and V_2 closed, then $Y = V_1$ or $Y = V_2$.

(1) Show that the set Spec(A) has minimal elements with respect to inclusion.
(2) Show that Spec(A) is irreducible if and only if $\mathrm{Nil}(A) \in \mathrm{Spec}(A)$.
(3) Let X be a topological space. Show that if Y is an irreducible subspace of X, then so is $\overline{Y}$.
(4) Show that every irreducible subspace of X is contained in a *maximal irreducible* subspace.
(5) Show that the *maximal irreducible* subspaces of X are closed and cover X.
(6) Find the *maximal irreducible* subspaces of Spec(A).

Problem 2. [10]

Let A be a commutative ring and let E, F be A-modules. We say that E is F-flat, if for any exact sequence $0 \longrightarrow F' \longrightarrow F$, the sequence $0 \longrightarrow E \otimes F' \longrightarrow E \otimes F$ is exact.

(1) Give an example for A, E, and F such that E is **not** F-flat.
(2) Let $\{F_i\}_i$ be a family of A-modules and F their direct sum. Prove: E is F_i-flat, $\forall\, i \Longleftrightarrow E$ is F-flat.
(3) Use (2) to prove that if $0 \longrightarrow E \otimes I \longrightarrow E$ is exact, for every ideal I of A, then E is flat.
(4) Use (3) to prove that the polynomial ring $A[X]$ is a flat A-algebra.

Problem 3. [10]

Let A be an integral domain with quotient field K and let M be an A-module. An element $x \in M$ is a torsion element of M if $\mathrm{Ann}_A(x) \neq 0$; that is, if x is killed by some non-zero element of A. The torsion elements of M form a sub-module of M, called the torsion submodule of M and denoted by $\mathrm{t}(M)$. If $\mathrm{t}(M) = 0$, the module M is said to be *torsion-free*. Prove:

(1) Each exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$ induces an exact sequence $0 \to \mathrm{t}(M') \xrightarrow{\overline{f}} \mathrm{t}(M) \xrightarrow{\overline{g}} \mathrm{t}(M'')$.
(2) If $g: M \to M''$ is onto, then $\overline{g}: \mathrm{t}(M) \to \mathrm{t}(M'')$ is not necessarily onto.
(3) Consider the mapping $\varphi: M \longrightarrow K \otimes M$, $x \mapsto 1 \otimes x$. Then $\mathrm{Ker}(\varphi) = \mathrm{t}(M)$.
(4) M is torsion free if and only if $M_{\mathfrak{m}}$ is torsion free for each $\mathfrak{m} \in \mathrm{Max}(A)$.
(5) Let I be an ideal of A. If $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \supseteq I$, then $M = IM$.

Problem 4. [15]

(1) Let A be a commutative ring. Show that the following are equivalent:
  (i) Every prime ideal of A is an intersection of maximal ideals;
  (ii) For every ideal I of A, the nilradical and Jacobson radical of A/I are equal;
  (iii) Every prime ideal of A, which is not maximal, is equal to the intersection of the prime ideals which contain it strictly.

  A ring A with any one of the three equivalent properties above is called a *Jacobson ring*.

(2) Let $A \subseteq B$ be an *integral* extension of commutative rings. Prove that if A is Jacobson, then so is B.

(3) Give an example of a commutative ring which is Jacobson and another example which is NOT.

Problem 5. [15]

(1) Let k be a field and $A := k[x_1, \ldots, x_n]$ be a finitely generated k-algebra. Let $k(x_1, \ldots, x_n) := \mathrm{qf}(A)$ and let r denote the transcendence degree of $k(x_1, \ldots, x_n)$ over k. By [Hungerford, VI–Corollary 1.7], $\{x_1, \ldots, x_n\}$ contains a transcendence base of $k(x_1, \ldots, x_n)$ over k; that is, $n \geq r$. Throughout, assume $n \gneqq r$. Prove:

  (a) $\sum_{(i_1,\ldots,i_n)\in\Delta} a_{(i_1,\ldots,i_n)}\, x_1^{i_1} \cdots x_n^{i_n} = 0$, for some finite set $\Delta \subset \mathbb{N}^n$ and nonzero elements $a_{(i_1,\ldots,i_n)}$'s in k.

  (b) Let N be a positive integer such that $N \gneqq \sup\{i_1, \ldots, i_n\}$, for all $(i_1, \ldots, i_n) \in \Delta$. Then, for $(i_1, \ldots, i_n) \in \Delta$ and $(j_1, \ldots, j_n) \in \Delta$, we have:
$$i_1 + Ni_2 + \cdots + N^{n-1} i_n = j_1 + Nj_2 + \cdots + N^{n-1} j_n \Rightarrow (i_1, \ldots, i_n) = (j_1, \ldots, j_n).$$

  (c) Let $y_i = x_i - x_1^{N^{i-1}}$, for all $i \in \{2, \ldots, n\}$. Then, there exist a nonzero positive integer m, a nonzero element $a \in k$, and a *polynomial* $f \in k[\mathbf{X_1}, \ldots, \mathbf{X_n}]$ with $\deg_{\mathbf{X_1}}(f) \leq m - 1$ such that:
$$ax_1^m + f(x_1, y_2, \ldots, y_n) = 0.$$

  (d) $k[y_2, \ldots, y_n] \subseteq A$ is an integral extension.

  (e) There exist $t_1, \ldots, t_r \in A$, *algebraically independent* over k, such that the extension $k[t_1, \ldots, t_r] \subseteq A$ is integral (*Noether's Normalization Lemma*). (The case $n = r$ is trivial.)

(2) Let A be a subring of an integral domain B such that B is a finitely generated A-algebra. Then, there exist $0 \neq a$ in A and $y_1, \ldots, y_n$ in B algebraically independent over A such that the ring extension $A[y_1, \ldots, y_n][1/a] \subseteq B[1/a]$ is integral.

Problem 6. [20]

(1) Let R be a Noetherian local ring with maximal ideal M and let $d(R)$ denote the dimension of M/M^2 as an R/M-vector space. Prove that, for each $x \in M \setminus M^2$, $d(R) = d(R/xR) + 1$.

(2) Let R be a Noetherian ring and let $X_1, \ldots, X_n$ be indeterminates over R. Let $p \in \mathrm{Spec}(R)$. Prove that $d(R[X_1, \ldots, X_n]_{p[X_1,\ldots,X_n]}) = d(R_p)$.
