King Fahd University of Petroleum and Minerals

MX-Program

Math557 – 231 Exam 2

Time Allowed: 150 minutes

Name:________________________________ ID#____________________ Sec#__________

Q#	Marks Obtained	Total Marks
1		12
2		26
3		12
4		14
5		12
6		12
7		12
Total		100

Q1. (12 points) Compute eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ -1 & 1 & 2 \end{bmatrix}$.

Q2. (26 points) Eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are 1, 3, 3. Answer following questions:

(a) (6 points) Find eigenvectors A.

(b) (2 points) Is this a defective or nondefective matrix.

(c) (2 points) What is the rank of this matrix.

(d) (2 points) Write Algebraic multiplicity M_a and Geometric multiplicity M_g of each eigenvalue.

(e) (2 points) Is the matrix A diagonalizable?

(f) (4 points) Verify Rayleigh Quotient for each eigenvector.

(g) (4 points) Find a unitary matrix U that diagonalizes A and write $A = UDU^{-1}$.

(h) (4 points) Is the matrix A normal or not? Show your work.

Q3. (12 points) Find TWO Schur factorizations of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. Verify your answer by multiplication of factors.

Q4. (10+2+2 points) Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) Find SVD and SVF of

(b) Compute $\|A\|_F$ by two different ways.

(c) Compute the spectral norm $\|A\|_2$.

Q5. (12 points) Consider the following data to fit a least square quadratic polynomial,

$$P_2(t) = x_1 + x_2 t + x_3 t^2.$$

t_i	0	1	2	3	4
y_i	1	3	4	8	12

(a) Write the matrix A and the column b needed to solve $Ax = b$.

(b) Write normal equations to find unknown vector x. Do not solve the normal equations.

(c) Write steps to find the unknown vector x using SVD approach.

Q6. (12 points) Consider the system,

$$3x_1 - x_2 + x_3 = 1$$
$$3x_1 + 6x_2 + 2x_3 = 0$$
$$3x_1 + 3x_2 + 7x_3 = 4$$

Express the Jacobi iteration method for the linear system $Ax = b$ in the form,

$$x^{(k)} = T_J x^{(k-1)} + C_J, \quad k = 1, 2, \cdots$$

and find first two iterations with initial guess $x^{(0)} = (0,\ 0,\ 0,)^T$.

Write a stopping criterion to stop the iterations.

Q7. (12 points) Consider $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, with $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

and eigenvalues od A are $3.4142,\ 2.0,\ 0.5858$. Verify the inequality,

$$\frac{1}{K_2(A)} \frac{\|e\|_2}{\|b\|_2} \leq \frac{\|y - x\|_2}{\|x\|_2} \leq K_2(A) \frac{\|e\|_2}{\|b\|_2}$$

for the perturbed system $Ax = b + e$ with $b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $and\ e = \begin{bmatrix} 0.005 \\ 0.0002 \\ -0.005 \end{bmatrix}$

Formula Sheet for Math557 Exam 1

1. Iterative Methods

$$x_i^{(k)} = \frac{1}{a_{ii}}\left(-\sum_{j=1}^{i-1} a_{ij}x_j^{(k-1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + b_i\right), \quad k = 1, 2, 3, \cdots.$$

$$x_i^{(k)} = \frac{1}{a_{ii}}\left(-\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + b_i\right), \quad k = 1, 2, 3, \cdots.$$

$$x_i^{(k)} = \frac{\omega}{a_{ii}}\left(-\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + b_i\right) + (1-\omega)x_i^{(k-1)}, 0 < \omega < 2.$$

2. Matrix form of Iterative Methods

$$X^{(k)} = D^{-1}(L+U)X^{(k-1)} + D^{-1}b, \quad k = 1, 2, \cdots$$

$$X^{(k)} = (D-L)^{-1}UX^{(k-1)} + (D-L)^{-1}b, \quad k = 1, 2, \cdots$$

3. Spectral Norm

$\|A\|_2 = \sigma_1$ and $\|A^{-1}\|_2 = \dfrac{1}{\sigma_n}$.

$\|A\|_2 = \lambda_1$ and $\|A^{-1}\|_2 = \dfrac{1}{\lambda_n}$, if A is positive definite.

$\|A\|_2 = |\lambda_1|$ and $\|A^{-1}\|_2 = \dfrac{1}{|\lambda_n|}$, if A is normal.

4. Spectral Condition Number

$$K_2(A) = \begin{cases} \dfrac{\lambda_1}{\lambda_n}, & if\ A\ is\ postivie\ definite, \\ \dfrac{|\lambda_1|}{|\lambda_n|}, & if\ A\ is\ normal, \\ \dfrac{\sigma_1}{\sigma_n}, & in\ general. \end{cases}$$

5. The Raleigh quotient of a vector x is the scalar,

$$R(x) = \frac{x^*Ax}{x^*x}$$