King Fahd University of Petroleum and Minerals

MX-Program

Math557 – 231 Final

Time Allowed: 180 minutes

Name:________________________________ ID#___________________ Sec#_________

Q#	Marks Obtained	Total Marks
1		15
2		15
3		16
4		12
5		10
6		15
7		10
8		12
9		15
Total		120

Q1. (5+5+5) Consider the matrix $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$.

(a) Use quadratic form of A and show that A is a positive definite matrix.

(b) Find LDL^* factorization of A.

(c) Find Cholesky factorization of A.

Q2. (15 points) Consider the matrix,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$$

(a) (8 points) Find eigenvalues and eigenvectors A.

(b) (2 points) Is this a defective or nondefective matrix.

(c) (2 points) Write Algebraic multiplicity and Geometric multiplicity of each eigenvalue.

(d) (3 points) Verify Rayleigh Quotient for each eigenvector.

Q3. (12+2+2 points) (a) Find SVD and SVF of $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$.

(b) Compute $\|A\|_F$ by two different ways.

(c) Compute the spectral norm $\|A\|_2$.

Q4. (12 points) Consider the system,

$$3x_1 - x_2 + x_3 = 1$$
$$3x_1 + 6x_2 + 2x_3 = 0$$
$$3x_1 + 3x_2 + 7x_3 = 4$$

Find first two iterations of the Gauss-Seidel method with initial guess $x^{(0)} = (0,\ 0,\ 0,)^T$.

Q5. (10 points) Let $x = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Find the Householder transformations H∈ $R^{3\times3}$ such that,

$$Hx = \|x\|_2 e_1$$

Verify your answer.

Q6. (10+5 points) Find QR-factorization of $A = \begin{bmatrix} 1 & 0 \\ -2 & -1 \\ 2 & 2 \end{bmatrix}$ using Gram-Scmidt orthogonalization.

Then solve the system $Ax = b$ using QR-factorization, where $b = [1, 2, 3]^T$.

Q7. (10 points) Consider the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$, write and draw all the **Gershgorin's circles.** Write the smallest possible interval that contains all the eigenvalues. Compute eigenvalues of this matrix and determine if they lie inside that interval. Can this matrix be singular?

Q8. (12 points) Consider the system $Ax = b,$ where $A = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix},$ $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Using conjugate gradient method with $x^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to find first two iterations, that is, x^1, r_1, β_1, p_1.

Q9. (15 points) Compute x^1, λ^1 and x^2, λ^2 for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $x^0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to approximate the largest eigenvalue and corresponding eigenvector using the power method. If $\lambda = 5$ is the largest eigenvalue of A, compute the absolute error ϵ and error bound δ for $q = \lambda^2$ using x^2.

Formula Sheet for Math557 Final Exam

1. LU factorization form

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_1 & d_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-2} & d_{n-1} & c_{n-1} \\ & & & a_{n-1} & d_n \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ l_1 & 1 & & & \\ & \ddots & \ddots & \ddots & \\ & & l_{n-2} & 1 & \\ & & & l_{n-1} & 1 \end{bmatrix} \begin{bmatrix} u_1 & c_1 & & & \\ & u_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & u_{n-1} & c_{n-1} \\ & & & & u_n \end{bmatrix}$$

$$u_1 = d_1, \qquad l_k = \frac{a_k}{u_k}, \qquad u_{k+1} = d_{k+1} - l_k c_k, \qquad k = 1, 2, \ldots, n-1.$$

2. The LDL* factorization of Hermitian *A*

$$A = \begin{bmatrix} a & \bar{b} \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} 1 & \bar{l}_1 \\ 0 & 1 \end{bmatrix}$$

$$d_1 = a, \qquad a l_1 = b \;\; or \;\; l_1 = \frac{b}{a}, \qquad d_2 = d - a|l_1|^2.$$

3. The Cholesky factorization (LL* factorization)

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} d_1 & 0 \\ l_1 & d_2 \end{bmatrix} \begin{bmatrix} d_1 & l_1 \\ 0 & d_2 \end{bmatrix}$$

$$d_1 = \sqrt{a}, \quad d_1 l_1 = b \;\Longrightarrow\; l_1 = \frac{b}{\sqrt{a}}, \quad l_1^2 + d_2^2 = c \;\Longrightarrow\; d_2 = \sqrt{c - l_1^2}$$

4. Quadratic form and Raleigh quotient $A \in \mathbb{C}^{n\times n}$

$$Q(x) = x^* A x, \qquad R(x) = \frac{x^* A x}{x^* x}, \qquad A \in \mathbb{C}^{n\times n}, \qquad x \in \mathbb{R}^n$$

Theorem 4.3 (Necessary Conditions for Positive (Semi)Definiteness) *If $A \in \mathbb{C}^{n\times n}$ is positive (semi)definite then for all i, j with $i \neq j$*

1. $a_{ii} > 0$, $(a_{ii} \geq 0)$,
2. $|Re\,(a_{ij})| < (a_{ii} + a_{jj})/2$, $(|Re\,(a_{ij})| \leq (a_{ii} + a_{jj})/2)$,
3. $|a_{ij}| < \sqrt{a_{ii} a_{jj}}$, $(|a_{ij}| \leq \sqrt{a_{ii} a_{jj}})$,
4. *If **A** is positive semidefinite and $a_{ii} = 0$ for some i then $a_{ij} = a_{ji} = 0$ for $j = 1, \ldots, n$.*

5. Iterative Methods

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(-\sum_{j=1}^{i-1} a_{ij} x_j^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} + b_i \right), \quad k = 1, 2, 3, \cdots.$$

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(-\sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} + b_i \right), \quad k = 1, 2, 3, \cdots.$$

$$x_i^{(k)} = \frac{\omega}{a_{ii}} \left(-\sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} + b_i \right) + (1-\omega) x_i^{(k-1)}, 0 < \omega < 2.$$

6. Matrix form of Iterative Methods

$$X^{(k)} = D^{-1}(L+U)X^{(k-1)} + D^{-1}b, \quad k = 1, 2, \cdots$$

$$X^{(k)} = (D-L)^{-1}UX^{(k-1)} + (D-L)^{-1}b, \quad k = 1, 2, \cdots$$

7. Spectral Norm

$$\|A\|_2 = \sigma_1 \quad \text{and} \quad \|A^{-1}\|_2 = \frac{1}{\sigma_n}.$$

$$\|A\|_2 = \lambda_1 \quad \text{and} \quad \|A^{-1}\|_2 = \frac{1}{\lambda_n}, \qquad \text{if } A \text{ is positive definite.}$$

$$\|A\|_2 = |\lambda_1| \quad \text{and} \quad \|A^{-1}\|_2 = \frac{1}{|\lambda_n|}, \qquad \text{if } A \text{ is normal.}$$

8. Steepest Descent and Conjugate Gradient Algorithms

Chooes x^0 $r_0 = b - Ax^0$ For $k = 0, 1, \ldots$ $p_k = r_k$ $t_k = Ap_k$ $\alpha_k = \frac{p_k^T r_k}{p_k^T t_k}$ $x^{k+1} = x^k + \alpha_k p_k$, $r_{k+1} = r_k - \alpha_k t_k$	Given initial guess x^0 $p_0 = r_0 = b - Ax_0$ For $k = 0,1,2, \ldots$ $t_k = Ap_k$ $\alpha_k = \frac{r_k^T r_k}{p_k^T t_k}$ $x^{k+1} = x^k + \alpha_k p_k$, $r_{k+1} = r_k - \alpha_k t_k$ $\beta_k = \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k}$, $p_{k+1} = r_{k+1} + \beta_k p_k$

9. Power and Inverse Power Algorithms

Given A, and maximum number of iterations kmax Given $x^0 \neq 0$, Tol For $k = 0 :$ kmax $z^{k+1} = Ax^k$ $x^{k+1} = \frac{z^{k+1}}{\|z^{k+1}\|}$ $\lambda = \left(x^{k+1}\right)^T A\, x^{k+1}$ if $\left\|Ax^{k+1} - \lambda x^{k+1}\right\| < Tol$, stop End $\epsilon = \lambda - q$, $\delta = \sqrt{\frac{(x^2)^T A^T A x^2}{(x^2)^T x^2} - q^2}$	Given A, and maximum numer of iterations kmax Given $x^0 \neq 0, Tol$ **For** $k = 0 :$ kmax Solve $A\, z^{k+1} = x^k$ $x^{k+1} = \frac{z^{k+1}}{\|z^{k+1}\|}$ $\mu = \left(z^{k+1}\right)^T A z^{k+1} = \left(z^{k+1}\right)^T x^k$ $\lambda = \frac{1}{\mu}$ if $\left\|Ax^{k+1} - \lambda x^{k+1}\right\| < Tol$, stop End

10. Gram-Schmidt Orthogonalization and QR-Factorization: Given $A = [a_1, a_2, a_3, \cdots, a_n]$

$$v_1 := a_1, \qquad v_j := a_j - \sum_{i=1}^{j-1} \frac{\langle a_j, v_i\rangle}{\langle v_i, v_i\rangle}\ v_i, \qquad j = 2, \ldots, n.$$

$Q := [q_1,\ q_2, \cdots,\ q_n]$ where $q_j := \frac{v_j}{\|v_j\|_2}$, $j = 1, 2, \cdots, n$

$$R = \begin{bmatrix} \|v_1\|_2 & a_2^T q_1 & a_3^T q_1 & \cdots & a_{n-1}^T q_1 & a_n^T q_1 \\ 0 & \|v_2\|_2 & a_3^T q_2 & \cdots & a_{n-1}^T q_2 & a_n^T q_2 \\ 0 & 0 & \|v_3\|_2 & \cdots & a_{n-1}^T q_3 & a_n^T q_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \|v_{n-1}\|_2 & a_n^T q_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \|v_n\|_2 \end{bmatrix}, \quad A = QR$$

11. Householder Transformation: Given x, find H

$$v = x \mp \|x\|_2 e_1, \qquad u = \frac{v}{\|v\|_2}, \qquad H = I - 2u \cdot u^T$$