MIDTERM

Duration: 90 minutes

ID:	
NAME:	Solution

- Show your work.
- Use the space provided to answer the question. If the space is not enough, continue on the back of the page.

Problem	Score
1	
2	
3	
4	
5	
6	
Total	/80
Score	/20

Problem 1 (10 points)

Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Find

(a) $\text{rank}(A) = \dim(\mathcal{R}(A))$ (rank of A)

(b) $\text{null}(A) = \dim(\mathcal{N}(A))$ (nullity of A)

Solution:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{R_1+R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\text{rank}(A) = 2$ and $\text{null}(A) = 3 - 2 = 1$.

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Problem 2 (10 points)

Consider the matrices

$$M = \begin{bmatrix} A & A^2B \\ O & B \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} I \\ O \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, O is the 2×2 zero matrix, I is the 2×2 identity matrix and B is a nonsingular 2×2 matrix. Compute $\mathcal{C}^T M^{-1}$.

Solution:

$$\left[\begin{array}{cc|cc} A & A^2B & I & O \\ O & B & O & I \end{array}\right] \xrightarrow[B^{-1}R_2]{A^{-1}R_1} \left[\begin{array}{cc|cc} I & AB & A^{-1} & O \\ O & I & O & B^{-1} \end{array}\right] \xrightarrow{-ABR_2+R_1} \left[\begin{array}{cc|cc} I & 0 & A^{-1} & -A \\ O & I & O & B^{-1} \end{array}\right]$$

Therefore

$$M^{-1} \begin{bmatrix} A^{-1} & -A \\ O & B^{-1} \end{bmatrix}$$

and we have

$$C^T M^{-1} = \begin{bmatrix} A^{-1} & -A \end{bmatrix} = \begin{bmatrix} -3 & 2 & -1 & -2 \\ 2 & -1 & -2 & -3 \end{bmatrix}$$

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Problem 3 (20 points)

Use LU factorization to solve the following system

$$Tx = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

where $T = \text{tridiag}_4(1, 2, 1)$ and $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

Solution:

$$T = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1+R_2} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{-\frac{2}{3}R_2+R_3} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{-\frac{3}{4}R_3+R_4}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

So

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

and

$$LUx = b = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

Let $y = Ux$, then $Ly = b$ and by forward substitution, we obtain

$$y = \begin{bmatrix} 2 \\ 2 \\ 5/3 \\ 3/4 \end{bmatrix}$$

Finally, we solve $Ux = y$ by backward substitution

$$x = \begin{bmatrix} 3/5 \\ 4/5 \\ 4/5 \\ 3/5 \end{bmatrix}$$

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Problem 4 (20 points)

Consider the following matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Using Gaussian elimination, find the following if possible; if not, state the reason why.

(a) LU factorization of A (that is $L1U$ factorization).

(b) $LU1$ factorization of A.

(c) LDU factorization of A.

(d) LDL^* factorization of A.

Solution:

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[-\frac{1}{2}R_1+R_3]{\frac{1}{2}R_1+R_2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

So

(a) $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $A = LU1 = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$

(d) LDL^* not possible; since A is not symmetric.

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Problem 5 (10 points)

Find Cholesky factorization of the matrix

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 21 \end{bmatrix}$$

Solution:

A=

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 21 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1 + R_3]{-\frac{1}{2}R_1 + R_2} \begin{bmatrix} 4 & 2 & -2 \\ 0 & 1 & 2 \\ 0 & 2 & 20 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 4 & 2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{bmatrix}$$

So, we have

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

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Problem 6 (10 points)

Find a suitable Housedholder transformation H and a scalar α such that

$$Hx = \alpha e_1$$

where $x = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 2 \end{bmatrix}$ and $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Solution:
Let $(x_1, x_2, x_3, x_4) = x = (4, 1, -2, 2)$ and define $\rho = \frac{x_1}{|x_1|} = \frac{4}{4} = 1$. Then

$$\alpha = -\rho\|x\| = -\sqrt{4^2 + 1^1 + (-2)^2 + 2^2} = -\sqrt{25} = -5.$$

Now let $z = (z_1, z_2, z_3, z_4) = \overline{\rho}x/\|x\| = \frac{1}{5}(4, 1, -2, 2)$. Finally define

$$u = \frac{z + e_1}{\sqrt{1 + z_1}} = \frac{1}{\sqrt{1 + 4/5}}(1 + 4/5, 1/5, -2/5, 2/5) = \frac{\sqrt{5}}{15}(9, 1, -2, 2).$$

and

$$\begin{aligned} H &= I - uu^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{45}\begin{bmatrix} 9 \\ 1 \\ -2 \\ 2 \end{bmatrix}\begin{bmatrix} 9 & 1 & -2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{45}\begin{bmatrix} 81 & 9 & -18 & 18 \\ 9 & 1 & -2 & 2 \\ -18 & -2 & 4 & -4 \\ 18 & 2 & -4 & 4 \end{bmatrix} = \begin{bmatrix} -4/5 & -1/5 & 2/5 & -2/5 \\ -1/5 & 44/45 & 2/45 & -2/45 \\ 2/5 & 2/45 & 41/45 & 4/45 \\ -2/5 & -2/45 & 4/45 & 41/45 \end{bmatrix} \end{aligned}$$

Note

$$Hx = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha e_1.$$

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