FINAL EXAM

Duration: 180 minutes

ID:	
NAME:	Solution

- Show your work.
- Use the space provided to answer the question. If the space is not enough, continue on the back of the page.

Problem	Score
1	/10
2	/10
3	/10
4	/20
5	/20
6	/20
7	/15
8	/15
Total	/120
Score	/30

Problem 1 (10 points)

Let

$$A = \begin{bmatrix} 4 & -2 & 6 \\ -2 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

Given that $\lambda = 4$ is an eigenvalue of A with algebraic multiplicity equal to 3, find the Jordan Factorization of A.

Solution:

First we find the eigenspace corresponding to $\lambda = 4$ by solving $(A - 4I)v = 0$ where $v = [v_1, v_2, v_3]^T$. To this end, we use Gaussian elimination

$$A - 4I = \begin{bmatrix} 0 & -2 & 6 \\ -2 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & 1 \\ 0 & -2 & 6 \end{bmatrix} \xrightarrow{2R_1 + R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & -2 & 6 \end{bmatrix} \xrightarrow{-2R_2 + R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $v_2 = 3v_3$ and $v_1 = v_2 - v_3$ and hence the eigenspace is generated by the vector $v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. This shows that $g_A(\lambda) = 1$ and A is defective with defect equal to 2. To find the Jordan factorization J of A, we solve $(A - 4I)^3 w = 0$ which is $\mathbf{0}w = 0$ and choose $w = [1, 0, 0]^T$. The first generalized eigenvector will be labeled

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_2 = (A - 4I)\mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1 = (A - 4I)\mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$$

Finally,

$$A = SJS^{-1} \quad \text{where} \quad S = [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3] = \begin{bmatrix} -2 & 0 & 1 \\ -3 & -2 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -3 \\ 1 & -2 & 4 \end{bmatrix}$$

and

$$J = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

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Problem 2 (10 points)

Use Gram-Schmidt orthogonalization of the columns of A to find the unique QR factorization of A where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Solution:
Let $A = [\mathbf{a}_1 \;\; \mathbf{a}_2]$

$$\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \mathbf{a}_2 - \frac{\langle \mathbf{v}_1, \mathbf{a}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}.$$

Now let

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

and $Q = [\mathbf{q}_1 \quad \mathbf{q}_2]$ and

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2}\frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{2}\frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix}$$

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Problem 3 (10 points)

Find the singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution:

Suppose that $A = U\Sigma V^T$ is the svd of A, then $A^TA = V\Sigma^T\Sigma V^T$. So we find the eigenvalues of $B = A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, by solving $|B - \lambda I| = 0$;

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0 \quad \Rightarrow \lambda - 2 = \pm 1 \Rightarrow \lambda = 3, 1.$$

For $\lambda = 3$, we have

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \overset{R_1+R_1}{\longrightarrow} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

So, $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a normal eigenvector corresponding to $\lambda = 3$.

For $\lambda = 1$, we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \overset{-R_1+R_1}{\longrightarrow} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So, $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a normal eigenvector corresponding to $\lambda = 3$. Then

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \text{and} \quad \sigma_1 = \sqrt{3},\ \sigma_2 = 1.$$

Now, we compute $U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$ by

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

To find a $\mathbf{u}_3$, let $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then we use Gram-Schmidt to find

$$\mathbf{u}_3 = v_3 - \langle \mathbf{u}_1, \mathbf{v}_3 \rangle \mathbf{u}_1 - \langle \mathbf{u}_2, \mathbf{v}_3 \rangle \mathbf{u}_2 = \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Finally, we have

$$A = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{3} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{3} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

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Problem 4 (20 points)

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Find

(a) $\|A\|_F$. (Frobenius norm)

(b) $\|A\|_1$. (One norm)

(c) $\|A\|_2$. (Spectral Norm)

(d) $K_2(A)$. (Spectral condition number)

Solution:

(a) $\|A\|_F = \sqrt{2^2 + 1^2 + 1^2 + 2^2} = \sqrt{10}$

(b) $\|A\|_1 = \max \left\{ \left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\|_1, \left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\|_1 \right\} = 3$

(c) $\|A\|_2$. Since A is positive definite, $\|A\|_2 = \lambda_1 = 3$ see previous problem.

(d) $K_2(A) = \|A\|_2 \|A^{-1}\|_2 = 3 * (1/1) = 3$.

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Problem 5 (20 points)

Let $b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ and $A = U\Sigma V^*$ where

$$U = \begin{bmatrix} -0.53 & -0.83 & 1.0 \\ 0.8 & -0.55 & 0.0 \\ 0.27 & 0.0 & 0.0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1.9 & 0.0 \\ 0.0 & 1.8 \\ 0.0 & 0.0 \end{bmatrix}, \quad V = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

(a) Find A^+; that is the generalized inverse of A.

(b) Use A^+ to find the least square solution with minimal Euclidean norm of the system $Ax = b$.

Solution:

(a)

$$\begin{aligned} A^+ &= V\Sigma_1^{-1} * U_1^T = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} 1/1.9 & 0.0 \\ 0.0 & 1/1.8 \end{bmatrix} \begin{bmatrix} -0.53 & 0.8 & 0.27 \\ -0.83 & -0.55 & 0.0 \end{bmatrix} \\ &= \begin{bmatrix} -0.278947 & 0.421053 & 0.142105 \\ -0.461111 & -0.305556 & 0.0 \end{bmatrix} \end{aligned}$$

(b)

$$\begin{aligned} Ax &= b \\ x &= A^+b \\ x &= \begin{bmatrix} -0.278947 & 0.421053 & 0.142105 \\ -0.461111 & -0.305556 & 0.0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.284211 \\ -0.766667 \end{bmatrix} \end{aligned}$$

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Problem 6 (20 points)

Consider the system

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0. \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{aligned}$$

Then

(a) Write the splitting matrices M_J and M_{SOR} for the Jacobi's method and Successive overrelaxation method (SOR method) respectively.

(b) Use M_J to write a fixed-point iteration for Jacobi's method.

(c) starting from $x_0 = \mathbf{0}$, use the fixed-point iteration in part (b), to find the first iterate x_1.

Solution:

Let

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

Define

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad A_L = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ -3 & -3 & 0 \end{bmatrix}.$$

Then

(a) $M_J = D$ and $M_{\text{SOR}} = \frac{1}{\omega}D - A_L$ for $\omega \in (0, 2)$.

(b) $x_{k+1} = (I - D^{-1}A)x_k + D^{-1}b$.

(c)

$$x_1 = D^{-1}b = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \\ 4/7 \end{bmatrix}$$

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Problem 7 (15 points)

Perform two steps of the conjugate gradient method to approximate a solution to the system

$$\begin{aligned} 3x_1 - x_2 &= 1 \\ -x_1 + 6x_2 &= 0 \end{aligned}$$

Solution:

Staring with $x_0 = \mathbf{0}$ (any x_0 will do). Then $p_0 = r_0 = b - Ax = b$ and

(**Iteration** 1) Compute $\alpha_0 = \frac{\langle r_0, r_0 \rangle}{\langle Ap_0, p_0 \rangle} = \frac{1}{3}$ and update

$$\begin{aligned} x_1 &= x_0 + \alpha_0 p_0 &= \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}, \\ r_1 &= r_0 - \alpha_0 A p_0 &= \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}, \\ \beta_0 &= \frac{\langle r_1, r_1 \rangle}{\langle r_0, r_0 \rangle} &= \frac{1}{9}, \\ p_1 &= r_1 + \beta_0 p_0 &= \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \end{bmatrix} \end{aligned}$$

(**Iteration** 2) Compute $\alpha_1 = \frac{\langle r_1, r_1 \rangle}{\langle Ap_1, p_1 \rangle} = \frac{1/9}{17/27} = \frac{3}{17}$ and update

$$\begin{aligned} x_2 &= x_1 + \alpha_1 p_1 &= \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} + \frac{3}{17} \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \end{bmatrix} &= \begin{bmatrix} \frac{6}{17} \\ \frac{1}{17} \end{bmatrix}, \\ r_2 &= r_1 - \alpha_1 A p_1 &= \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} - \frac{3}{17} \begin{bmatrix} 0 \\ \frac{17}{9} \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned}$$

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Problem 8 (15 points)

Let

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}.$$

Starting from the vector $z_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) Apply two iterations of the power method to approximate the dominant eigenvector of A.

(b) Approximate the dominant eigenvalue of A using the Rayleigh quotient formula.

Solution:

(a) Let $x_0 = z_0/\|z_0\|_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

(**Iteration** 1) $z_1 = Ax_0 = \begin{bmatrix} 2.83 \\ -2.12 \end{bmatrix}$, then $x_1 = z_1/\|z_1\|_2 = \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}$

(**Iteration** 2) $z_2 = Ax_a = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, then $x_2 = z_2/\|z_2\|_2 = \begin{bmatrix} 0.83 \\ -0.55 \end{bmatrix}$

(**Iteration** 3) $\lambda_1 = \langle Ax_2, x_2 \rangle = 3.62$.

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