MATH557: Applied Linear Algebra
Midterm Exam (Term-251)

18th October 2025[a]

[a] *Semester 251*

Instructions

- Show all essential steps. Exact values are preferred; keep radicals as needed.
- You may state (without proof) standard theorems about eigen-decomposition, Cholesky, SVD.
- Unless specified, vectors are columns. Use clear notation for bases and orthogonality.
- Total: **40 points**.

Problem 1 — Diagonalization (8 points)

Consider

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) **Find** an invertible matrix X and a diagonal matrix Λ such that

$$A = X\Lambda X^{-1}. \qquad \text{(4 pts)}$$

(b) Using (a), **explain** how to obtain Q and R with $A^n = QRQ^{-1}$ for any integer $n \geq 1$. (2 pts)

(c) **Compute** A^2 and A^3 **explicitly** via your factorization. (2 pts)

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Problem 2 — Cholesky Factorization (8 points)

Let

$$S = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(a) **Determine** whether a Cholesky factorization exists (i.e., $S = LL^\top$ with L lower-triangular and positive diagonal). **Justify** via leading principal minors (positive definiteness). (3 pts)

(b) If it exists, **compute** the Cholesky factor L explicitly. (5 pts)

Problem 3 — Nonnegative Matrix Factorization (8 points)

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Starting with $U^{\mathsf{T}} = (1, 0)$ and enforcing $U, V \geq 0$, **perform** *three full iterations:2 necessarily, 1 optional* of the alternating-minimization scheme to approximately solve

$$\min_{U,V \geq 0} \|A - UV\|_F^2.$$

(5 pts)

(b) Briefly **interpret** how the iterates reflect a low-rank, nonnegative approximation of A. (You may keep intermediate expressions simple and omit line-by-line algebraic simplification.) (3 pts)

Hint: You can use your answer to **Problem 1-a** in your interpretation.

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Problem 4 — Spectral Factorization (8 points)

For

$$S = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix},$$

(a) **Find** the eigenvalues and a corresponding set of eigenvectors. (3 pts)

(b) **Assemble** an orthonormal matrix Q and diagonal matrix Λ so that

$$S = Q\Lambda Q^{\mathsf{T}}. \qquad \text{(3 pts)}$$

(c) **Explain** how Q and Λ relate to the quadratic form $x^{\mathsf{T}}Sx$. (2 pts)

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Problem 5 — Singular Value Decomposition (8 points)

Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) **Compute** an SVD of $A = U\Sigma V^{\mathsf{T}}$. Keep square roots as needed; avoid heavy symbolic simplification. (4 pts)

Hint: Work via $A^t A$ to get V and the singular values.

(b) **Express** A as the sum of rank-one matrices using your SVD. (2 pts)

(c) **Identify** the closest rank-one approximation to A in Frobenius norm and **write it explicitly**. (2 pts)

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