Math559: Numerical Linear Algebra
Midterm Exam

28th October 2024 at 7:00pm[1]

[a] *Duration 120 minutes*

NAME:

KFUPM ID:

Question 1 (8 points)

a. For the given matrix

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix},$$

find X and Λ such that the diagonalization of $A = X\Lambda X^{-1}$.

b. Show that if $Ax = \lambda x$ and $Ay = \Lambda y$, then $A(x+y) = \lambda(x+y)$.

c. Illustrate the statement in (b) on the two eigenvalues of A.

Question 2 (5 points)

Given the matrix:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

find the inverse of $M = A - uv^t$, where $u^t = (1\ 1)$ and $v^t = (2\ 1)$.

N.B.: You have to use the SMW formula seen in class.

Question 3 (5 points)

Consider the matrix

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 5 \end{pmatrix}.$$

a. Starting with an initial $U_0^t = (1, 1)$, detail the first full iterations that would solve:

$$minimize \, \|A - UV\|_F^2 \,, \text{ with } U, V \geq 0.$$

b. What is the rank-one matrix A_1 approximating A reached?

Question 4 (10 points)

a. Find the pseudo-inverse matrix A^+ for the given matrix:

$$A = \begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}.$$

b. For $b^t = (4\ 2)$, **show** that $x^+ = A^+ b$ is the least squares solution to $Ax = b$.

Question 5 (5 points)

If two matrices A and B are factorized using GSVD where

$$U_a = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad U_b = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}, \quad \Sigma_a = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}, \quad \Sigma_b = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix},$$

$$\text{and} \quad Z = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix},$$

then $\sqrt{3}(A + B) = ?$

Question 6 (7 points)

Consider the matrix A such that:

$$A = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}.$$

a. Perform manually the Gram-Schmidt factorization $A = QR$.

b. Verify that your solution satisfies the mathematical conditions:

$$A^t A = R^t R \text{ and } A^{-1} = R^{-1} Q^t.$$