MATH559: Numerical Linear Algebra
Final Exam

23rd December 2024 at 7:00pm[a]

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[b] *Duration: 180 minutes*

Question 1 (10 points)

Consider the matrix

$$A(t) = \begin{pmatrix} 4t & 7t \\ 2 & 4 \end{pmatrix}.$$

a. Find the derivative of its inverse matrix: $\frac{\partial A^{-1}}{\partial t}$.

b. Use the formula seen in class to verify your result in **(a)**.

c. Find the derivatives of the eigenvalues of A: $\frac{\partial \lambda}{\partial t}$.

. **Question 2 (10 points)**

Perform the Cholesky Factorization for $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

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Question 3 (10 points)

Perform the Gram-Schmidt factorization of the matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

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Question 4 (10 points)

a) Find the matrix $S = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$, with the smallest l_2 norm $||S||_2$ and such that:

$$S \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

b) Find the best possible real-valued matrix A^* that completes $A = \begin{pmatrix} * & 1 \\ 3 & * \end{pmatrix}$ and minimizes:

$$\min_{C \text{ and } R} \frac{1}{2}||(A - CR^t)_{known}||_2^2 + \frac{1}{2}(||C||_F^2 + ||R||_F^2)$$

where $CR^t = (2 \times 1)(1 \times 2)$.

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. **Question 5 (15 points)**

Consider the matrices $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 3 & s & 0 \\ 0 & t & 2 \end{pmatrix}$

and the vector $B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, where s and t are real-valued parameters.

a. Find the least squares solution X^* to the system: $AX = B$.

b. Find and **Verify** the relation between A and D^* the solution to the system:

$$DA^+B = B.$$

. **Question 6 (15 points)**

Consider the problem where we have to:

$$\text{minimize } f(x_1, x_2) = (x_1 + 1)^2 + (x_2 - 1)^4 + x_2^2 + 2x_1x_2$$

a. Perform manually one full iteration of the Gradient Descent (GD) method starting from the point $(0, 0)$. Is your obtained point optimal?

b. Perform manually one full iteration of the Newton-Raphson (NR) method starting from the point $(0, 0)$. Is your obtained point optimal?

c. Perform manually one full iteration of the Generalized Inverse Newton-Raphson (GINR) method starting from the point $(1, 1)$. Is your obtained point optimal?

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. **Question 7 (15 points)**

In a simple neural network, we have T observations of two inputs p_1 and p_2, and a unique neuron in the output layer forecasting the value of y.

Assume that:

$$\hat{y} = f(w_1, w_2, w_3, b) = w_1 p_1 p_2 + w_2 p_1 + w_3 p_2 + b.$$

a) Find the general expression of the error function gradient vector ∇E.

b) Find the general expression of the error function Hessian matrix H.

c) Find H when $T = 1000$, and
$\sum_{t=1}^{T} p_1^t = 2000$, $\sum_{t=1}^{T} p_2^t = 2500$, $\sum_{t=1}^{T} (p_1^t)^2 = 40000$, $\sum_{t=1}^{T} (p_2^t)^2 = 50000$,
$\sum_{t=1}^{T} (p_1^t)(p_2^t) = 50000$, $\sum_{t=1}^{T} (p_1^t)^2 (p_2^t) = 60000$, $\sum_{t=1}^{T} p_1^t (p_2^t)^2 = 80000$,
$\sum_{t=1}^{T} (p_1^t)^2 (p_2^t)^2 = 100000$.

d) Use your answer in **(b)** and **(c)** to explain whether the Hessian matrix has any interesting property that could be useful to compute its inverse.

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. **Question 8 (15 points)**

Consider the matrix $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

a. Perform a spectral factorization for the matrix A.

b. Without computation, use your understanding of the course to explain how your answer in **(a)** helps getting the Singular Vector Decomposition of A.

c. Use your answer in **(a)** to get A^{-1} the inverse matrix of A.

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