King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 564
Midterm Exam — Term 211
Tuesday, November 09, 2021
Allowed Time: 75 minutes

Instructor: **Dr. Boubaker Smii**

Name: ____________________________________

ID #: ________________________

Section #: ____________ Serial Number: ____________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justifications !

Question #	Grade	Maximum Points
1		10
2		9
3		15
4		13
5		13
Total:		60

Exercise 1:(10)

A multiple choice Exam is composed of 10 questions, each question consists of 4 possible answers such that only one is true. The student chooses randomly an answer for each question. Find the probability that:

1- He/ She has exactly 5 correct answers.

Let X be the random variable denoting the number of Correct answers. Thus $X \sim Bin(n=10, p=\frac{1}{4})$, the Binomial distibution of X is: $P(X=k)=\binom{10}{k}\left(\frac{1}{4}\right)^k\left(1-\frac{1}{4}\right)^{10-k}$,

03

Therefore $P(X=5)=\binom{10}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^5=0{,}058.$

02

2- He/ She has at most 9 correct answers.

We have:

$$P(X\leq 9)=1-P(X=10)=1-\left(\frac{1}{4}\right)^{10}$$

$$=0.999\ldots\simeq 1$$

02

3- He/ She has at least 2 correct answers.

$$P(X\geq 2)=1-P(X<2)=1-P(X\leq 1)$$

$$=1-P(X=0)-P(X=1)$$

$$=1-\binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10-0}-\binom{10}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^9$$

$$=0{,}756$$

03

Exercise 2:(9)

Assume that arrivals of customers into a store follow a Poisson process with rate $\lambda = 30$ arrivals per hour. Suppose that the probability that a customer buys something is $p = 0.30$.

1- Find the expected number of sales made during an eight-hour business day.

Let $N_1(t)$ be the number of arrivals who buy something.
Let $N_2(t)$ be the number of arrivals who do not buy something.

05 N_1 and N_2 are two independent Poisson processes.

The rate for N_1 is $\lambda_1 = \lambda p = 30 \times 0.3 = 9$

The rate for N_2 is $\lambda_2 = \lambda(1-p) = 30 \times 0.7 = 21$

Therefore $E(N_1) = \lambda_1 \times t = 9 \times 8 = 72$

2-Find the probability that 15 or more sales are made in one hour.

$$P(N_1 \geq 15) = 1 - \sum_{i=0}^{14} P(N_1 = i)$$

$$= 1 - \sum_{i=0}^{14} e^{-\lambda_1} \frac{\lambda_1^i}{i!}$$

04

$$= 1 - \sum_{i=0}^{14} e^{-9} \frac{9^i}{i!}$$

Exercise 3:(15)

Consider the Markov chain $\{X_n, n \geq 0\}$ with three states, $S = 1, 2, 3$, that has the following transition matrix

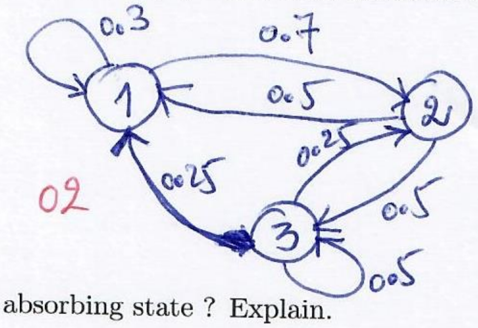
$$P = \begin{pmatrix} 0.3 & 0.7 & a \\ 0.5 & b & 0.5 \\ c & 0.25 & 0.5 \end{pmatrix};$$

1. Find the real numbers a, b, c and draw the state transition diagram for this chain.

$a = 0$

$b = 0$ 01

$c = 0.25$



$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.3 & 0.7 & 0 \\ 2 & 0.5 & 0 & 0.5 \\ 3 & 0.25 & 0.25 & 0.5 \end{array}$$

2. Is there any absorbing state ? Explain.

02 No, all states are accessible.

3. Determine $P(X_2 = 3 \mid X_1 = 3)$.

02 $P(X_2 = 3 \mid X_1 = 3) = P_{33} = 0.5$

4. Assume that $P(X_0 = 1) = 0.5$, determine $P(X_1 = 2, X_2 = 3, X_3 = 1 \mid X_0 = 1)$.

$P(X_1 = 2, X_2 = 3, X_3 = 1 \mid X_0 = 1) = P(X_0 = 1) P(X_1 = 2 \mid X_0 = 1) P(X_2 = 3 \mid X_1 = 2) P(X_3 = 1 \mid X_2 = 3)$

03

$= 0.5 \times P_{12} P_{23} P_{31} = 0.5 \times 0.7 \times 0.5 \times 0.25$

$= 0.043$

5. Determine $P(X_{12} = 1 \mid X_5 = 3, X_{10} = 1)$.

02 $P(X_{12} = 1 \mid X_5 = 3, X_{10} = 1) = P(X_{12} = 1 \mid X_{10} = 1) = P_{11}^{(2)} = 0.44$

6. Determine $P(X_3 = 3,\ X_5 = 1 \mid X_1 = 1)$.

03 $P(X_3 = 3, X_5 = 1 \mid X_1 = 1) = P(X_3 = 3 \mid X_1 = 1) P(X_5 = 1 \mid X_3 = 3)$

$= P_{13}^{(2)} \times P_{31}^{(2)} = 0.35 \times 0.325 = 0.113$

Exercise 4: (13)

Consider the standard Brownian motion $\{B_t,\ t \geq 0\}$.

A)- 1- Verify that $C_t = t\,B_{\frac{1}{t}},\ t > 0;\ C_0 = 0,$ is also a Brownian motion.

$*\ C_0 = 0$ and $t \mapsto C_t$ Continuous and $C_t - C_s \hookrightarrow N(0, t-s)$, $0<t<s$

$*\ Var(C_t - C_s) = t - s$ and $Var(C_{t-s}) = Var\left((t-s)\,B_{\frac{1}{t-s}}\right) = (t-s)^2 \times \frac{1}{t-s} = t-s.$

$*$ Finally the increment of C_t are independent since, for $0 < u \leq s < t$: $Cov(C_u, C_t - C_s) = Cov(C_u, C_t) - Cov(C_u, C_s)$

$$Cov(C_u, C_t - C_s) = Cov\left(u.B_{\frac{1}{u}}, t\,B_{\frac{1}{t}}\right) - Cov\left(u.B_{\frac{1}{u}}, s\,B_{\frac{1}{s}}\right)$$
$$= ut\min\left(\tfrac{1}{u}, \tfrac{1}{t}\right) - us\min\left(\tfrac{1}{u}, \tfrac{1}{s}\right)$$
$$= ut.\tfrac{1}{t} - us.\tfrac{1}{s} = u - u = 0$$

Hence $C_t, t>0$ is also a Brownian motion.

04

2- Let $X_t = B_t - t\,B_1,\ 0 \leq t \leq 1,$ be the Brownian Bridge.
Find $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$.

$*\ E(X_t) = E(B_t - tB_1) = E(B_t) - E(tB_1) = 0 - 0 = 0$ (02)

$*\ Cov(X_t, X_s) = E(X_tX_s) - E(X_t)E(X_s)$

$$= E[(B_t - tB_1)(B_s - sB_1)] - E(B_t - tB_1)E(B_s - sB_1)$$
$$= E(B_tB_s - sB_tB_1 - tB_1B_s + stB_1^2) - 0$$
$$= \min(s,t) - st$$

(03)

Therefore $Var(X_t) = t - t^2 = t(1-t)$; Hence $\begin{cases} E(X_t) = 0 \\ Var(X_t) = t(1-t). \end{cases}$

B)- Let $\{Y_t,\ t \geq 0\}$ be a stochastic process which satisfies: $Y_t = 1 + 0.2\,t + 0.5\,B_t$.
Find $\mathbf{P}[Y_{10} > 1 \mid Y_0 = 1]$.
(Express the result as $\Phi(x)$, where Φ is the standard normal distribution function and x a real number.)

$*$ We Know that $B_t \hookrightarrow N(0, t)$

$*\ Y_t - Y_0 = 1 + 0.2t + 0.5B_t - 1 = 0.2t + 0.5B_t$, Therefore

$\left.\begin{array}{l} E(Y_t - Y_0) = 0.2t \\ Var(Y_t - Y_0) = (0.5)^2 t \end{array}\right\}$ $Y_{10} - Y_0 \hookrightarrow N(2, 2.5)$, Thus

$Y_t - Y_0 \hookrightarrow N((0.2)t, (0.5)^2 t)$

$*\ P(Y_{10} > 1 \mid Y_0 = 1) = P(Y_{10} - Y_0 > 0 \mid Y_0 = 1) = P(Y_{10} - Y_0 > 0)$ (04)

$$= \Phi\left(\frac{-2}{\sqrt{2.5}}\right)$$

Exercise 5: (13)

Consider the geometric Brownian motion given by

$$X_t = e^{\mu t + \sigma B_t}, \ t \geq 0, \ \sigma > 0, \ \mu \in \mathbb{R}. \qquad \text{(a)}$$

A)- Give an application where we can use the geometric Brownian motion X_t.

The Geometric Brownian motion can be
(03) used in Quantitative Finance. It is used for modelling the stock-prices in the Black-sholes model.

B)- We assume that the price $X_t, t \geq 0$ of a risky asset (called stock) at time t is given by a geometric Brownian motion of the form:

$$X_t = X_0 \, e^{\mu t + \sigma B_t}, \quad (\mu = c - \frac{1}{2}\sigma^2), \quad t \geq 0, \ \sigma > 0, \ \mu \in \mathbb{R}. \qquad \text{(b)}$$

1- What represent the quantity $c\,dt + \sigma\,dB_t$ during the period time $[t, t + dt]$?

During the period $[t, t+dt]$, the quantity $c\,dt + \sigma\,dB_t$
(03) is the relative return from the asset.

2- Give a brief description of the following :

a- $\sigma\,dB_t$:

(02) stochastic noise.

b- The constant c:

(02) mean rate of return

c- The constant $\sigma > 0$:

(03) the volatility, it measure the riskiness of the asset.