King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 564
Final Exam — Term 211
Wednesday, December 29, 2021
Allowed Time: 100 minutes

Instructor: **Dr. Boubaker Smii**

Name: ____

ID #: ____________________________

Section #: _______________ Serial Number: _______________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justifications !

Question #	Grade	Maximum Points
1		**10**
2		**10**
3		**10**
4		**10**
5		**10**
6		**8**
7		**12**
8		**10**
Total:		**80**

Exercise 1:(10)

An insurance company believes that people can be devided into tow classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a person who is not accident prone.

1- If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy ?

2- Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone ?

Exercise 2:(10)

A- Let X be a random variable with probability density function (pdf):

$$f(x) = x\, e^{-\frac{x^2}{2}}, \;\; 0 < x < \infty. \tag{a}$$

1- Find the cumulative distribution function (CDF) of X and Y.

2- Find the probability density of $Y = X^2$.

B- A discrete random variable, Y , has probability mass function

$$P_Y(y) = c\,(y-3)^2, \;\; y = -2, -1, 0, 1, 2. \tag{b}$$

1. Find the value of the constant c.

Solution:

2. Give the cumulative distribution function of Y .

Solution:

3. Find the mean (Expected value) and variance of Y.

Exercise 3:(10)

Consider the standard Brownian motion $\{B_t,\, t \geq 0\}$.

A- Explain why the classical integration fails to integrate the Brownian sample paths.

B- Give two major differences between the Riemann and Itô integrals.

C- Let f be a smooth function and a, b constants such that $a < b$.

Given that $I = \displaystyle\int_a^b f(t)\, dB_t$ is the Itô stochastic integral, what represent the quantity: $\frac{dB_t}{dt}$?

Exercise 4:(10)
Let $\{B_t,\, t \geq 0\}$ be a standard Brownian motion .

1- Use the Itô formula to find an expression of B_t^2.

2- Find $\displaystyle\int_0^t B_s^2\, dB_s$.

Exercise 5: (10)
Let $\{B_t,\, t \geq 0\}$ be a Brownian motion .

A)- i- Let $f(t,x) = e^{x-\frac{1}{2}t}$. Use the extended Itô formula to express the function $f(t,B_t)$ in terms of Itô stochastic integrals .

ii- What is called the function $f(t,B_t)$?

B)- Consider a particular form of the Geometric Brownian motion given by:

$$X_t = e^{(c-\frac{1}{2}\sigma^2)t+\sigma B_t}, \;\; c,\, \sigma > 0. \qquad \text{(c)}$$

1- Apply the Itô formula to the Geometric Brownian motion given in equation (c) and verify that the stochastic process X_t satisfies a given equation.

2- What is called the equation obtained in the previous question 1- ?

Exercise 6:(8)
Verify that the process $X_t = \frac{B_t}{1+t}$ solves the stochastic differential equation:

$$dX_t = -\frac{1}{1+t} X_t \, dt + \frac{1}{1+t} \, dB_t; \quad X_0 = 0 \tag{d}$$

Exercise 7:(12)
A- The mean reverting Ornstein-Uhlenbeck process is the solution X_t of the stochastic differential equation:

$$dX_t = (m - X_t)\, dt + \sigma\, dB_t, \tag{e}$$

where m, σ are real constants and $B_t \in \mathbb{R}$.
1- Find the solution X_t of the stochastic differential equation (e).
Hint: You may apply the Itô formula to the function $f(t, x) = e^t\, x$.

2-Find $\mathbb{E}(X_t)$.

C.(5) The Black-Scholes-Merton model for growth with uncertain rate of return, is the value of §1 after time t, invested in a saving account. It is described by the following stochastic differential equation:

$$dX_t = \mu\, X_t\, dt + \ \sigma\, X_t\, dB_t, \ \ \mu, \sigma > 0 \tag{f}$$

1- Give the type of the SDE (f).

2- Use the Itô formula to verify that the solution of the SDE (f) is given by a Geometric Brownian motion.

Exercise 8:(10)
A- Assume that the value of a **portfolio** (wealth) at time t is given by:

$$V_t = a_t\, X_t + b_t\, A_t$$

(You want to hold certain amounts of shares: a_t in stock and b_t in bond).
Describe the following situation:

- $a_t < 0$:

- $b_t < 0$:

B- For $s < t$ we can buy (or sell) shares of the stock at price $X(s)$ and then sell (or buy) these shares at time t for the price $X(t)$.

Suppose that the option gives us the right to buy one share of the stock at time t for a price K.
1- Give the expression of the worth of option at time t.

2- Describe the **European** and **American** call option.

C- Let $X_t = X_0 + c\int_0^t X_s\, ds + \sigma\int_0^t X_s\, dB_s$.
The **Black-Scholes** option pricing formula is given by

$$\begin{aligned} V_0 &= u(T, X_0) \\ &= X_0\, \Phi(g(T, X_0)) - K e^{-rT}\, \Phi(h(T, X_0)), \end{aligned}$$

which is a rational price at time $t = 0$ for an European option with exercise price K.
1- Find the value of your self-financing portfolio at time $t \in [0, T]$.

2- The value of the self-financing portfolio is **independent** on which <u>coefficient</u> ?