

King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 565 Final Exam
The First Semester of 2022-2023 (221)
Time Allowed: 150mn

Name:

ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1: Consider the nonlinear system

$$\frac{dx}{dt} = (x-y)(x+1)$$

$$\frac{dy}{dt} = (x+y)(y-1).$$

- 1.) (8pts) Find all critical points of the system.
- 2.) (12pts) Study stability of the critical points of the system.

Solution

$$1) \begin{cases} (x-y)(x+1) = 0 \\ (x+y)(y-1) = 0 \end{cases} \Rightarrow x=y \text{ or } x=-1 \Rightarrow (y-1)^2 = 0, y=1$$

$$\quad \quad \quad \neq \quad 2y(y-1) = 0, y=0, 1$$

The critical points are the origin, $A\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $B\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

2.) Stability by linearization

The Jacobian $J(x_0, y_0) = \begin{pmatrix} x_0+1+x_0-y_0 & -(x_0+1) \\ y_0-1 & y_0-1+x_0+y_0 \end{pmatrix}$

At A,

$$J = \begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix}, \quad \begin{vmatrix} 2-\lambda & -2 \\ 0 & 2-\lambda \end{vmatrix} = 0, \quad (2-\lambda)^2 = 0, \quad \lambda = 2, 2$$

A is unstable

At the origin:

$$J = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad \begin{vmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix} = 0, \quad -(1-\lambda^2) - 1 = 0, \quad \lambda = \pm\sqrt{2}$$

∴ The origin is unstable

At B

$$J = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{vmatrix} -2-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0, \quad \lambda(2+\lambda) = 0, \quad \lambda = 0, -2$$

B is stable.

Problem 2: Consider the nonlinear system

$$\begin{aligned}\frac{dx}{dt} &= x + x^2 - xy + y^2 + f(x, y) \\ \frac{dy}{dt} &= 2x - y + xy.\end{aligned}$$

Assume that

$$f(0,0) = 0 \quad \text{and} \quad \lim_{(x,y) \rightarrow (0,0)} \frac{|f(x,y)|}{|x| + |y|} = 0.$$

1.) (14pts) Show that the system is almost linear.

2.) (6pts) Deduce the stability of the origin.

Solution

$$1.) \quad \text{If } x = \begin{pmatrix} x \\ y \end{pmatrix}, \text{ then } \dot{x} = \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}}_A x + \underbrace{\begin{pmatrix} x^2 - xy + y^2 + f(x,y) \\ xy \end{pmatrix}}_{F(x)}$$

$$\begin{aligned}\text{We have } |F(x)| &= |x^2 - xy + y^2 + f(x,y)| + |xy| \\ &\leq \underbrace{x^2 + |xy| + y^2}_{\leq \frac{1}{2}x^2 + \frac{1}{2}y^2} + |f(x,y)| + \underbrace{|x||y|}_{\leq \frac{1}{2}x^2 + \frac{1}{2}y^2}\end{aligned}$$

$$\begin{aligned}&\leq 2(x^2 + y^2) + |f(x,y)| \\ &\leq 2(|x| + |y|)^2 + |f(x,y)| \\ \Rightarrow \frac{|F(x)|}{|x|} &\leq 2(|x| + |y|) + \frac{|f(x,y)|}{|x| + |y|}\end{aligned}$$

$$\Rightarrow \lim_{|x| \rightarrow 0} \frac{|F(x)|}{|x|} = 0, \text{ since limit of the RHS is zero}$$

$$2.) \quad \begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} = 0, \quad -(1-\lambda^2) = 0, \quad \lambda = \pm 1$$

The origin is unstable.

Problem 3: Consider the linear system

$$\begin{cases} x' = \left(2 + \frac{2 \cos 2t}{3 + \sin 2t}\right) x \\ y' = x - y. \end{cases} \quad (1)$$

1.) (4pts) Write System (1) in the form $X' = A(t)X$, where $X = \begin{pmatrix} x \\ y \end{pmatrix}$. What is the smallest period of $A(t)$?

2.) (8pts) Show that the general solution (1) is

$$\begin{aligned} x(t) &= c_1(3 + \sin 2t)e^{2t}, \\ y(t) &= c_1\left(1 + \frac{3}{13} \sin 2t - \frac{2}{13} \cos 2t\right)e^{2t} + c_2e^{-t} \end{aligned}$$

where c_1, c_2 are arbitrary constants.

You may use the relation $\int e^{3t} \sin 2t dt = \left(\frac{3}{13} \sin 2t - \frac{2}{13} \cos 2t\right)e^{3t}$.

3.) (8pts) Find the characteristic multipliers ρ_1 and ρ_2 of the system.

1.)
$$\dot{X} = \underbrace{\begin{pmatrix} 2 + \frac{2 \cos 2t}{3 + \sin 2t} & 0 \\ 1 & -1 \end{pmatrix}}_{A(t)} X. \quad \begin{aligned} &A(t+\pi) = A(t). \\ &A \text{ is periodic of period } \pi \end{aligned}$$

2.)
$$\int \frac{dx}{x} = \int \left(2 + \frac{2 \cos 2t}{3 + \sin 2t}\right) dt \Rightarrow \ln|x| = 2t + \ln(3 + \sin 2t) + C$$

$$\Rightarrow x(t) = c_1(3 + \sin 2t)e^{2t}$$

$$\frac{dy}{dt} + y = x, \quad \frac{d}{dt}(ye^t) = xe^t = c_1(3 + \sin 2t)e^{3t}$$

$$\Rightarrow ye^t = c_1 \int (3 + \sin 2t)e^{3t} dt + c_2$$

$$= c_1 \left[e^{3t} + \int \sin 2t e^{3t} dt \right] + c_2$$

$$= c_1 \left[1 + \frac{3}{13} \sin 2t - \frac{2}{13} \cos 2t \right] e^{3t} + c_2$$

$$\Rightarrow y(t) = c_1 \left[1 + \frac{3}{13} \sin 2t - \frac{2}{13} \cos 2t \right] e^{2t} + c_2 e^{-t}$$

3.)
$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \underbrace{\begin{pmatrix} 3 + \sin 2t \\ 1 + \frac{3}{13} \sin 2t - \frac{2}{13} \cos 2t \end{pmatrix}}_{X_1} e^{2t} + c_2 \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{X_2} e^{-t}$$

X_1 and X_2 are solutions of the system.

$\lambda_1 = 2, \lambda_2 = -1$ are characteristic exponents

$\rho_1 = e^{2\pi}, \rho_2 = e^{-\pi}$ are characteristic multipliers

Problem 4: Consider the linear system

$$\begin{aligned}\frac{dx}{dt} &= -x - 4y \\ \frac{dy}{dt} &= 4x.\end{aligned}$$

Let $V(x, y) = x^2 + y^2$ and $W(x, y) = x^2 + \varepsilon xy + y^2$

1.) (6pts) Compute $\frac{dV}{dt}$. Deduce the stability of the origin.

2.) (14pts) Compute $\frac{dW}{dt}$. Find a value of $\varepsilon > 0$ such that $\frac{dW}{dt}$ is negative definite and W is positive definite on \mathbb{R}^2 . Deduce the stability of the origin.

Solution

$$1.) \quad \frac{dV}{dt} = 2x\dot{x} + 2y\dot{y} = 2x(-x-4y) + 2y(4x) = -2x^2 = -V^*$$

V is positive definite on \mathbb{R}^2 , V^* is negative
 \Rightarrow the origin is stable.

$$\begin{aligned}2.) \quad \frac{dW}{dt} &= 2x\dot{x} + \varepsilon(x\dot{y} + \dot{x}y) + 2y\dot{y} \\ &= 2x(-x-4y) + \varepsilon[4x^2 + y(-x-4y)] + 8yx \\ &= -(2-4\varepsilon)x^2 - \varepsilon xy - 4\varepsilon y^2\end{aligned}$$

$$\begin{aligned}\Delta &= \varepsilon^2 - 16\varepsilon(2-4\varepsilon) \\ &= \varepsilon[\varepsilon - 16(2-4\varepsilon)] = \varepsilon[65\varepsilon - 32]\end{aligned}$$

if $\varepsilon \in (0, \frac{32}{65})$, then $\Delta < 0$, $\frac{dW}{dt} < 0$

For W , $\Delta = \varepsilon^2 - 4$,

if $\varepsilon \in (0, 2)$, then $W > 0$

Conclusion: If we choose $\varepsilon \in (0, \frac{32}{65})$,

then W is positive definite and $\frac{dW}{dt}$ is negative definite on \mathbb{R}^2 , thus the origin is globally asymptotically stable.

Problem 5:

1.) (10pts) Use Lyapunov second method to show that the origin is unstable for the nonlinear system

$$\begin{aligned}\frac{dx}{dt} &= x^3 - y^3 \\ \frac{dy}{dt} &= 3xy^2 + 6x^2y + 3y^3.\end{aligned}$$

2.) (10pts) Use Lyapunov second method to show that the origin is globally asymptotically stable for the nonlinear system

$$\begin{aligned}\frac{dx}{dt} &= -3x - 2xy^2 \\ \frac{dy}{dt} &= -2y - x^2y.\end{aligned}$$

Solution

$$1) \begin{cases} \frac{dx}{dt} = x^3 - y^3 \\ \frac{dy}{dt} = 3xy^2 + 6x^2y + 3y^3 \end{cases} \begin{matrix} \times x \\ \times y \end{matrix} \Rightarrow \begin{cases} \frac{1}{2} \frac{d}{dt} x^2 = x^4 - y^2x \\ \frac{1}{2} \frac{d}{dt} y^2 = 3xy^3 + 6x^2y^2 + 3y^4 \end{cases}$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} (3x^2 + y^2) = 3x^4 + 6x^2y^2 + 3y^4 = 3(x^4 + 2x^2y^2 + y^4)$$

$\underbrace{\hspace{10em}}_{\text{Positive definite on } \mathbb{R}^2}$
 $\underbrace{\hspace{10em}}_{(x^2+y^2)^2}$
 $\underbrace{\hspace{10em}}_{\text{Positive definite on } \mathbb{R}^2}$

\Rightarrow the origin is unstable

$$2) \begin{cases} \frac{dx}{dt} = -3x - 2xy^2 \\ \frac{dy}{dt} = -2y - x^2y \end{cases} \begin{matrix} \times x \\ \times y \end{matrix} \Rightarrow \begin{cases} \frac{1}{2} \frac{d}{dt} x^2 = -3x^2 - 2x^2y^2 \\ \frac{1}{2} \frac{d}{dt} y^2 = -2y^2 - x^2y^2 \end{cases}$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} (x^2 + y^2) = \underbrace{-3x^2 - 2y^2}_{\text{negative definite on } \mathbb{R}^2} - \underbrace{3x^2y^2}_{\text{negative on } \mathbb{R}^2}$$

$\underbrace{\hspace{10em}}_{\text{positive definite on } \mathbb{R}^2}$
 $\underbrace{\hspace{10em}}_{\text{negative on } \mathbb{R}^2}$
 $\underbrace{\hspace{10em}}_{\text{negative on } \mathbb{R}^2}$

\Rightarrow The origin is globally asymptotically stable.