King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 565 Midterm Exam
The First Semester of 2022-2023 (221)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1:

1.)(10pts) Find the explicit solution and its largest interval of definition of the IVP

$$\begin{cases} (x^2+1)\dfrac{dy}{dx} = x(y^2-1), \\ y(0) = 3. \end{cases}$$

2.)(10pts) Does the IVP

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{x^3-4}{x^2-y^2}, \\ y(0) = 1, \end{cases}$$

have a unique solution?

Solution:

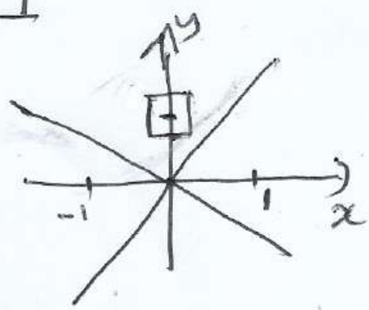
1.) $\dfrac{dy}{y^2-1} = \dfrac{x}{x^2+1}dx \Rightarrow \displaystyle\int\left(\frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}\right)dx = \int \frac{x}{x^2+1}dx$

$\dfrac{1}{2}\ln\left|\dfrac{y-1}{y+1}\right| = \dfrac{1}{2}\ln|x^2+1| + C \Rightarrow \dfrac{y-1}{y+1} = C(x^2+1)$

$y(0)=3 \Rightarrow \dfrac{2}{4} = C \Rightarrow \dfrac{y-1}{y+1} = \dfrac{1}{2}(x^2+1),$

Thus, $\boxed{y = \dfrac{x^2+3}{1-x^2},\ x \in (-1,1)}$

2.) $f(x,y) = \dfrac{x^3-4}{x^2-y^2}$



$D = \left\{(x,y) \in \mathbb{R}^2;\ \begin{matrix} |x| \le \frac{1}{4} \\ |y-1| \le \frac{1}{4} \end{matrix}\right\}$

$-\dfrac{1}{4} \le x \le \dfrac{1}{4} \Rightarrow -\dfrac{1}{4^3} \le x^3 \le \dfrac{1}{4^3} \Rightarrow -\dfrac{257}{64} \le x^3-4 \le -\dfrac{255}{64}$

$\Rightarrow \dfrac{255}{64} \le |x^3-4| \le \dfrac{257}{64}$

$\left.\begin{matrix} 0 \le x^2 \le \frac{1}{16} \\ \frac{3}{4} \le y \le \frac{5}{4} \end{matrix}\right\},\ \dfrac{9}{4} \le y^2 \le \dfrac{25}{4},\ -\dfrac{25}{4} \le -y^2 \le -\dfrac{9}{4},\ -\dfrac{25}{4} \le x^2-y^2 \le -\dfrac{35}{16}$

$\dfrac{35}{16} \le |x^2-y^2| \le \dfrac{25}{4};\quad |f(x,y)| = \dfrac{|x^3-4|}{|x^2-y^2|} \le \dfrac{257}{140}$

$\dfrac{\partial f}{\partial y} = \dfrac{-2y(x^3-4)}{(x^2-y^2)^2};\qquad \left|\dfrac{\partial f}{\partial y}\right| \le 2\cdot\dfrac{257}{245}$ on D

f is continuous on D, and $\dfrac{\partial f}{\partial y}$ is bounded

$\Rightarrow$ The IVP has a unique solution $y = y(x),\ x \in \left[\dfrac{-35}{257}, \dfrac{35}{257}\right]$

Problem 2: 1.)(10pts)Show that the solution of the initial value problem

$$\begin{cases} -\dfrac{dy^2}{dx} \leq \dfrac{1}{x}y^2 + \dfrac{e^{-x}}{x}, \\ y(0) = 20000, \end{cases}$$

satisfies

$$y^2(x) \leq \frac{e^{-x}-1}{x}, \quad \forall x<0.$$

2.)(10pts) If $f(y)$ is lipschitz, show that the solution of the IVP

$$\begin{cases} \dfrac{dy}{dx} = -y + f(y), \\ y(0) = -3, \end{cases}$$

is unique on $[0, \infty)$.

__Solution:__

1.) $-\dfrac{dy^2}{dx} \leq \dfrac{1}{x}y^2 + \dfrac{e^{-x}}{x} \Rightarrow \dfrac{dy^2}{dx} + \dfrac{1}{x}y^2 \geq -\dfrac{e^{-x}}{x}$

An integrating factor is $e^{\int \frac{dx}{x}} = e^{\ln|x|} = |x| = -x, \; x<0$

$\Rightarrow \dfrac{d}{dx}(-xy^2) \geq e^{-x} \Rightarrow \underbrace{\int_x^0 \dfrac{d}{dr}(-ry^2)\,dr}_{0+xy^2(x)} \geq \underbrace{\int_x^0 e^{-r}\,dr}_{e^{-x}-1}$

$\Rightarrow -xy^2(x) \geq e^{-x}-1 \Rightarrow y^2(x) \leq \dfrac{e^{-x}-1}{x}, \; \forall x<0$

2.) Let y_1 and y_2 be two solutions of the IVP

$\begin{cases} \dfrac{dy_1}{dx} = -y_1 + f(y_1) \\ y_1(0) = -3 \end{cases}$; $\begin{cases} \dfrac{dy_2}{dx} = -y_2 + f(y_2) \\ y_2(0) = -3 \end{cases}$ If $y = y_1 - y_2$, then $\Rightarrow \begin{cases} \dfrac{dy}{dx} = -y + f(y_1) - f(y_2) \\ y(0) = 0 \end{cases}$

$\Rightarrow \dfrac{d}{dx}(e^{x}y) = e^{x}(f(y_1) - f(y_2))$; $e^{x}y(x) = y(0) + \int_0^x e^r(f(y_1) - f(y_2))\,dr$

$\Rightarrow y(x) = y(0)e^{-x} + e^{-x}\int_0^x e^r(f(y_1) - f(y_2))\,dr$

$\Rightarrow |y(x)| \leq |y(0)|e^{-x} + e^{-x}\int_0^x e^r|f(y_1) - f(y_2)|\,dr$; $|f(y_1) - f(y_2)| \leq L|y|$

$\leq |y(0)| + \int_0^x L|y(r)|\,dr, \; \forall x \geq 0$

We apply the Gronwall inequality $\Rightarrow |y(x)| \leq \underbrace{|y(0)|}_{=0}e^{Lx}, \; x \geq 0$

$\Rightarrow |y(x)| = 0 \Rightarrow \boxed{y(x) = 0, \; \forall x \geq 0}$

Problem 3: Consider the nonhomogenous linear system

$$\frac{dx}{dt} = 4x - y - 3$$
$$\frac{dy}{dt} = x + 6y - 7.$$

1.)(4pts) Covert the system into a homogeneous linear system $X' = AX$.
2.)(8pts) Find a matrix P such that $X = PZ$ and $Z' = JZ$, where J is the Jordan matrix.
3.)(8pts) Draw the phase portrait of the system in the xy- plane.

Solution:

1.) Critical points: $\begin{cases} 4x - y = 3 \\ x + 6y = 7 \end{cases} \Rightarrow x = 1;\ y = 1;\ A\begin{pmatrix}1\\1\end{pmatrix}$

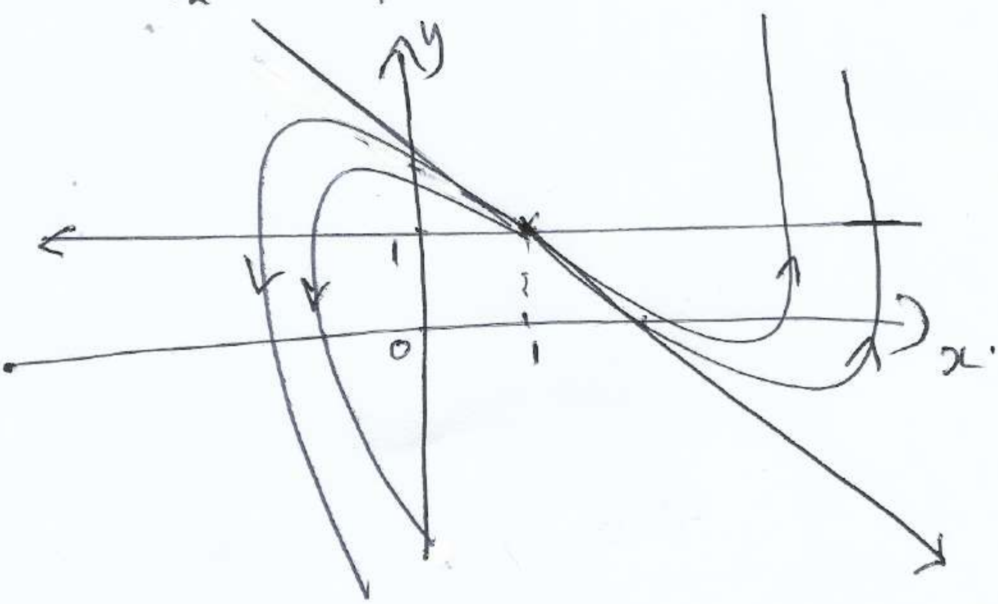
$$\left.\begin{matrix} u = x - 1 \\ v = y - 1 \end{matrix}\right\} \Rightarrow \begin{cases} \dfrac{du}{dt} = 4(u+1) - (v+1) - 3 = 4u - v \\ \dfrac{dv}{dt} = u + 1 + 6(v+1) - 7 = u + 6v \end{cases}$$

2.) $A = \begin{pmatrix} 4 & -1 \\ 1 & 6 \end{pmatrix}$ $\quad \begin{vmatrix} 4-\lambda & -1 \\ 1 & 6-\lambda \end{vmatrix} = 0;\ \lambda^2 - 10\lambda + 25 = 0,\ \underline{\lambda = 5, 5}$

$$\left.\begin{matrix} (A - 5I)K = 0;\ \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right),\ x + y = 0,\ K\begin{pmatrix}1\\-1\end{pmatrix} \\ (A - 5I)P = K,\ \left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right),\ x + y = -1,\ P\begin{pmatrix}-1\\0\end{pmatrix} \end{matrix}\right\} \Rightarrow P = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow Z' = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix} Z$$

3.) $\begin{matrix} z_1(t) = (\eta_1 + \eta_2 t)e^{5t} \\ z_2(t) = \eta_2 e^{5t} \end{matrix}$, if $Z(0) = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$



Problem 4: Consider the system

$$\frac{dx}{dt} = 2y^2 + 3x$$
$$\frac{dy}{dt} = y$$
$$\frac{dz}{dt} = y^2 - 2z.$$

1.)(4pts) What does the origin represent to the system? Indicate its nature.
2.)(16pts) Find the stable manifold of the system at the origin.

Solution:

1.) The origin is a critical point.

The Jacobian at (x_0, y_0, z_0) is $J = \begin{pmatrix} 3 & 4y_0 & 0 \\ 0 & 1 & 0 \\ 0 & 2y_0 & -2 \end{pmatrix}$

At the origin $J = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$; $\lambda = 3, 1, -2$

the origin is a saddle

2.) $\frac{dy}{dt} = y$, $\frac{d}{dt}(y e^{-t}) = 0$, $y e^{-t} = y_0$ $\Rightarrow$ $\boxed{y(t) = y_0 e^{t}}$

$\frac{dx}{dt} - 3x = 2y_0^2 e^{2t}$, $\frac{d}{dt}(x e^{-3t}) = 2y_0^2 e^{-t}$; $x e^{-3t} = x_0 + 2y_0^2 \underbrace{\int_0^t e^{-r} dr}_{1 - e^{-t}}$

$\Rightarrow x(t) = x_0 e^{3t} + 2y_0^2 e^{3t}(1 - e^{-t})$

$\boxed{x(t) = x_0 e^{3t} + 2y_0^2 e^{3t} - 2y_0^2 e^{2t}}$

$\frac{dz}{dt} + 2z = y_0^2 e^{2t}$; $\frac{d}{dt}(z e^{2t}) = y_0^2 e^{4t}$; $z e^{2t} = z_0 + y_0^2 \underbrace{\int_0^t e^{4r} dr}_{\frac{e^{4t} - 1}{4}}$

$\Rightarrow z(t) = z_0 e^{-2t} + y_0^2 \frac{e^{-2t}}{4}(e^{4t} - 1)$

$\boxed{z(t) = z_0 e^{-2t} - \frac{y_0^2}{4} e^{-2t} + \frac{y_0^2}{4} e^{2t}}$

Thus, $W^s(0,0) = \{(x, y, z) \in \mathbb{R}^3,\ x = 0, y = 0\}$: the z-axis

Problem 5:

Consider the nonlinear autonomous system

$$\frac{dx}{dt} = -x^5(3x^2 + 4y^2 - 2) - my$$
$$\frac{dy}{dt} = mx$$

for a nonzero real number m.

1) (4pts) Confirm that there is non negligible chance for the system to have a closed orbit.

2) (16 pts) Show that the system has at least one periodic solution.

Solution:

1.) $f(x,y) = -x^5(3x^2+4y^2-2) - my \Rightarrow f_x = -x^4(21x^2+20y^2-10)$

$g(x,y) = mx \Rightarrow g_y = 0$

$\nabla \cdot F = f_x + g_y = f_x$

$21x^2+20y^2-10=0$ (an ellipse)

From the Poincaré criteria, there might be a closed curve

2.) $\frac{1}{2}\frac{dx^2}{dt} = -x^6(3x^2+4y^2-2) - mxy$

$+\ \frac{1}{2}\frac{dy^2}{dt} = mxy$

$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = -x^6(2x^2+4y^2-2)$

Now, we notice that $2(x^2+y^2) \le 2x^2+4y^2 \le 4(x^2+y^2)$

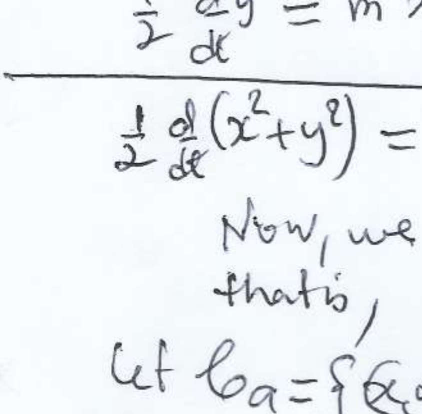
that is, $2(x^2+y^2-1) \le 2x^2+4y^2-2 \le 4(x^2+y^2-\frac{1}{2})$

Let $C_a = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 = a^2\}$

$C_b = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 = b^2\}$; $V(x,y) = \frac{1}{2}(x^2+y^2)$

$\Rightarrow \underbrace{-4x^6(x^2+y^2-\frac{1}{2})}_{\text{we want this } \ominus} \le \frac{dV}{dt} \le \underbrace{-2x^6(x^2+y^2-1)}_{\text{we want this } \oplus}$

$a^2 - \frac{1}{2} < 0$, $a < \frac{\sqrt{2}}{2}$ $\qquad$ $b^2 - 1 > 0$, $b > 1$



Let $D = \{(x,y) / \frac{1}{4} \le x^2+y^2 \le 4\}$

This is a trapping region.

(bounded and positively invariant)

$\xrightarrow[\text{Poincaré-Bendixson Theorem}]{}$ There is at least one periodic solution

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