King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 568 Final Exam
The Second Semester of 2022-2023 (222)
Time Allowed: 150mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1:
1.)(10pts) Solve the 2D Cauchy problem

$$u_{tt} = u_{xx} + u_{yy}, \quad (x,y) \in \mathbb{R}^2,\ t > 0, \tag{1}$$
$$u(x,y,0) = xy, \qquad (x,y) \in \mathbb{R}^2, \tag{2}$$
$$u_t(x,y,0) = 0, \qquad (x,y) \in \mathbb{R}^2. \tag{3}$$

2.)(10pts) the 3D Cauchy problem

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}, \quad (x,y,z) \in \mathbb{R}^3,\ t > 0, \tag{4}$$
$$u(x,y,z,0) = 0, \qquad (x,y,z) \in \mathbb{R}^3, \tag{5}$$
$$u_t(x,y,z,0) = z^2, \qquad (x,y,z) \in \mathbb{R}^3. \tag{6}$$

Justify your answers clearly.

Solution:

1.) The Kirchoff's formula gives $u(x,y,t) = \frac{\partial}{\partial t}(\frac{1}{4\pi t}\iint_{S_t} \xi\eta d\sigma_t)$, where S_t is the sphere of center $(x,y,0)$ ad radius t. First, we have $d\sigma_t = \frac{2t}{\sqrt{t^2-(\xi-x)^2-(\eta-y)^2}}d\xi d\eta$. Setting $\xi - x = r\cos\theta$ and $\eta - y = r\sin\theta$, $r \in [0,t]$ and $\theta \in [0,2\pi]$, we get that

$$\begin{aligned}
&\frac{1}{4\pi t}\iint_{S_t} \xi\eta d\sigma_t \\
&= \frac{1}{4\pi t}\int_0^t\int_0^{2\pi} \frac{2t}{\sqrt{t^2-r^2}}(y + r\sin\theta)(x + r\cos\theta) r dr d\theta \\
&= \frac{1}{2\pi}\Big(2\pi xy\int_0^t \frac{r}{\sqrt{t^2-r^2}}dr + \int_0^t \frac{r^2 x}{\sqrt{t^2-r^2}}dr \underbrace{\int_0^{2\pi}\sin\theta d\theta}_{=-[\cos\theta]_0^{2\pi}=0} + \int_0^t \frac{r^2 y}{\sqrt{t^2-r^2}}dr \underbrace{\int_0^{2\pi}\cos\theta d\theta}_{=[\sin\theta]_0^{2\pi}=0} \\
&\quad + \int_0^t \frac{r^3}{\sqrt{t^2-r^2}}dr \underbrace{\int_0^{2\pi}\cos\theta\sin\theta d\theta}_{=-\frac{1}{4}[\cos 2\theta]_0^{2\pi}=0}\Big) \\
&= xyt.
\end{aligned}$$

Thus, $u(x,y,t) = xy$

2.) The Kirchoff's formula gives $u(x,y,t) = \frac{1}{4\pi t}\iint_{S_t} \zeta^2 d\sigma_t$, where S_t is the sphere of center (x,y,z) ad radius t. Using spherical coordinates $\xi - x = t\sin\theta\cos\varphi$, $\eta - y = t\sin\theta\sin\varphi$, and $\zeta - z = t\cos\theta$, we have that $d\sigma_t = t^2\sin\theta d\theta d\varphi$. It follows that

$$\begin{aligned}
u(x,y,t) &= \frac{1}{4\pi t}\int_0^{2\pi}\int_0^{\pi} (z + t\cos\theta)^2 t^2 \sin\theta d\theta d\varphi \\
&= \frac{z^2 t}{2}\underbrace{\int_0^{\pi}\sin\theta d\theta}_{=-[\cos\theta]_0^{\pi}=2} + zt^2 \underbrace{\int_0^{\pi}\cos\theta\sin\theta d\theta}_{=\frac{1}{2}[\sin^2\theta]_0^{\pi}=0} + \frac{t^3}{2}\underbrace{\int_0^{\pi}\cos^2\theta\sin\theta d\theta}_{=-\frac{1}{3}[\cos^3\theta]_0^{\pi}=\frac{2}{3}} \\
&= z^2 t + \frac{t^3}{3}
\end{aligned}$$

Problem 2:
1.)(6pts) Consider a continuous solution u of the heat IBVP

$$\begin{cases} u_t = u_{xx}, \quad -1 < x < 1, \ t > 0 \\ u(x,0) = f(x), \quad -1 \leq x \leq 1, \\ u(-1,t) = g(t) - h(t), \ u(1,t) = -g(t), \quad t \geq 0, \end{cases} \tag{7}$$

on the rectangular domain $R = [-1,1] \times [0,1]$.
Explain clearly why the following assertion is true:
$|f(x)| \leq \varepsilon$, $|g(t)| \leq \varepsilon$, $|h(t)| \leq \varepsilon$, for $(x,t) \in R \Longrightarrow |u(x,t)| \leq 2\varepsilon$, for $(x,t) \in R$.

2.)(14pts) What is the maximum value of the continuous solution $v(x,t)$ to the IBVP

$$\begin{cases} v_t = v_{xx}, \quad 0 < x < 2, \ t > 0 \\ u(x,0) = \dfrac{x}{(1+x)^2}, \quad 0 \leq x \leq 2, \\ v(0,t) = \dfrac{1}{5}e^{-t}, \ v(2,t) = \dfrac{t}{6}, \quad t \geq 0. \end{cases} \tag{8}$$

on the rectangular domain $R = [0,2] \times [0,1]$?

<u>Solution:</u>
1.) We consider the lines: (L1): $x = -1$; (L2): $x = 1$; (L3): $t = 0$. We apply the weak maximum principle to v. Its says that v achieves its maximum value either on (L1), (L2) or (L3). On (L1) we have

$$v(x,t) \leq \max_{t \in [0,1]} |g(t) - h(t)| \leq 2\varepsilon,$$

as $|g(t) - h(t)| \leq |g(t)| + |h(t)| \leq 2\varepsilon$. On (L2) we have $v(x,t) \leq \max_{t\in[0,1]} |-g(t)| \leq \varepsilon$, as $|-g(t)| = |g(t)| \leq \varepsilon$. On (L3) we have $v(x,t) \leq \max_{x\in[-1,1]} |f(x)| \leq \varepsilon$, as $|f(x)| \leq \varepsilon$. Hence, $v(x,t) \leq 2\varepsilon$. We also apply the weak maximum principle to $-v(x,t)$, to get $-v(x,t) \leq 2\varepsilon$. It follows that $|v(x,t)| \leq 2\varepsilon$, for $(x,t) \in R$.

2.) We consider the lines: (L1): $x = 0$; (L2): $x = 2$; (L3): $t = 0$. We apply the weak maximum principle to v. Its says that v achieves its maximum value either on (L1), (L2) or (L3). On (L1) we have $v(x_1,t_1) \max_{t\in[0,1]}(\frac{1}{5}e^{-t}) = \frac{1}{5}$. On (L2) we have $v(x_2,t_2) = \max_{t\in[0,1]}(\frac{t}{6}) = \frac{1}{6}$. On (L3) we have $v(x_13, t_3) = \max_{x\in[0,2]} \frac{x}{(x+1)^2} = \frac{1}{4}$, as $f'(x) = \frac{1-x}{(x+1)^3}$ and $f(1) == \frac{1}{4}$ is the maximum value. Hence, $\max_{(x,t)\in R} v(x,t) = \frac{1}{4}$.

Problem 3:
1.)(10pts) Consider the boundary value problem

$$\begin{cases} y'' + \lambda y = 0, \\ y(0) - y(1) = 0 \\ y'(0) - y'(1) = 0. \end{cases}$$

Find values of λ and corresponding function $y(x)$ that are solutions to BVP.
2.) Consider the non-homogeneous BVP

$$\begin{cases} -y'' + 3y = f(x), \\ y'(0) = y'(\pi) = 0. \end{cases}$$

a.)(6pts) Find the $A_n \in \mathbb{R}$ such that $y(x) = \sum_{n=0}^{\infty} A_n \cos(nx)$ is solution to the BVP.
b.)(4pts) Compute the values of A_n for $f(x) = 1 + x$.

Solution:
The auxiliary equation is $m^2 + \lambda = 0$. Assume $\lambda = -\alpha^2$, $\alpha > 0$. Thus, $m^2 - \alpha^2 = 0$, and we deduce the solution $y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$. The boundary conditions imply that $c_1 = c_1 \cosh\alpha + c_2 \sinh\alpha$ and $c_2\alpha = c_1\alpha \sinh\alpha + c_2\alpha \cosh\alpha$, that is, $c_1 = c_2 = 0$. Now, assume that $\lambda = 0$. This implies that $y = c_1 x + c_2$, and the boundary conditions imply that $c_2 = c_1 + c_2$, that is, $c_1 = 0$. We get for the eigenvalue $\lambda_0 = 0$, the corresponding eigenfunction $y_0(x) = 1$. Lastly, we assume Assume $\lambda = \alpha^2$, $\alpha > 0$. Thus, $m^2 + \alpha^2 = 0$, and we deduce the solution $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. The boundary conditions imply that $c_1 = c_1 \cos\alpha + c_2 \sin\alpha$ and $c_2\alpha = -c_1\alpha \sin\alpha + c_2\alpha \cos\alpha$, that is, $\sin\alpha = 0$ & $\cos\alpha = 1$, or $\alpha = 2\pi n$. We then get the eigenvalues $\lambda_n = 4(\pi n)^2$ and corresponding eigenfunctions $y_n(x) = \cos(2\pi n x)$ and $\tilde{y}_n(x) = \sin(2\pi n x)$, for $n = 1, 2$.

2.) a.)We substitute y into the DE and after noting that $(\cos nx)'' = -n^2 \cos nx$, we find

$$3A_0 + \sum_{n=1}^{\infty} A_n(3 + n^2)\cos(nx) = f(x). \tag{9}$$

Thus, $3A_0\pi + \sum_{n=1}^{\infty} A_n(3+n^2) \underbrace{\int_0^\pi \cos(nx)dx}_{=\frac{1}{n}[\sin nx]_0^\pi = 0} = \int_0^\pi f(x)dx. \implies A_0 = \frac{1}{3\pi}\int_0^\pi f(x)dx.$

Now, we multiply (9) by $\cos mx$ $(m = 1, 2, ...)$, integrate between 0 and π, and we find

$$3A_0 \underbrace{\int_0^\pi \cos(mx)dx}_{=0} + \sum_{n=1}^{\infty} A_n(3+n^2) \underbrace{\int_0^\pi \cos(nx)\cos(mx)dx}_{=I_{nm}} = \int_0^\pi f(x)\cos(mx)dx.$$

Note that $\cos(nx)\cos(mx) = \frac{1}{2}[\cos(n+m)x + \cos(n-m)x] \implies I_{nm} = 0$, if $n \neq m$ & $= \frac{\pi}{2}$ if $n = m$. We infer that $\frac{\pi}{2}A_m(3 + m^2) = \int_0^\pi f(x)\cos(mx)dx \implies A_m = \frac{2}{(3+m^2)\pi}\int_0^\pi f(x)\cos(mx)dx$.
b.) If $f(x) = 1 + x$, then

$$A_0 = \frac{1}{3\pi}\int_0^\pi (1+x)dx = \frac{1}{3}(1 + \frac{\pi}{2})$$

and

$$A_m = \frac{2}{(3+m^2)\pi}\int_0^\pi (1+x)\cos(mx)dx = \frac{2((-1)^m - 1)}{m^2(3+m^2)}, \quad m = 1, 2, ...$$

Problem 4:

1.)(10pts) The normal derivative $\frac{\partial u}{\partial n}$ is equal to $\nabla u.n$, where n is the unit outer normal vector to the unit disk D. Convert $\frac{\partial u}{\partial n}$ into polar coordinates for $(x,y)\in D$. You can follow the following steps:

a.) Let $x=r\cos\theta$ and $y=r\sin\theta$. If we consider r as a function of θ, $r=r(\theta)$, then $n=\frac{\frac{dy}{d\theta}i-\frac{dx}{d\theta}j}{\sqrt{(\frac{dx}{d\theta})^2+(\frac{dy}{d\theta})^2}}$. Compute n in terms of r and θ.

b.) We know that, if $u(x,y)=v(r,\theta)$, then $\frac{\partial v}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$ and $\frac{\partial v}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$. Deduce $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in terms of r, θ and v.

c.) Use results from parts (a) and (b) to compute $\frac{\partial u}{\partial n}$.

2)(10pts) **Only explain clearly** how to construct a solution $u\in\mathcal{C}^2(\overline{\Omega})$ to the nonhomogeneous Neumann problem

$$\begin{cases}\Delta u=0, \ x\in\Omega,\\ \dfrac{\partial u}{\partial n}=g, \quad x\in\partial\Omega,\end{cases}$$

based on the green function, where Ω is a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$.

<u>**Solution:**</u>

1.)a.) If $x=r(\theta)\cos\theta$ and $y=r(\theta\sin\theta$, then $\frac{dx}{d\theta}=r'\cos\theta-r\sin\theta$ and $\frac{dy}{d\theta}=r'\sin\theta+r\cos\theta$. Thus, $\sqrt{(\frac{dx}{d\theta})^2+(\frac{dy}{d\theta})^2}=\sqrt{(r')^2+r^2}$, $n=\frac{1}{\sqrt{(r')^2+r^2}}[(r'\sin\theta+r\cos\theta)i-(r'\cos\theta-r\sin\theta)j]$.

b.)Now, $x=r\cos\theta$ and $y=r\sin\theta\Longrightarrow\frac{\partial v}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta$ and $\frac{\partial v}{\partial\theta}=-r\frac{\partial u}{\partial x}\sin\theta+r\frac{\partial u}{\partial y}\cos\theta$. We deduce that $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial v}{\partial\theta}\sin\theta$ and $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial v}{\partial\theta}\cos\theta$.

c.)It follows from part (a) and (b) that

$$\begin{aligned}\frac{\partial u}{\partial n}&=\frac{(r'\sin\theta+r\cos\theta)(\frac{\partial v}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial v}{\partial\theta}\sin\theta)-(r'\cos\theta-r\sin\theta)(\frac{\partial v}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial v}{\partial\theta}\cos\theta)}{\sqrt{(r')^2+r^2}}\\&=\frac{r\frac{\partial v}{\partial r}-\frac{r'}{r}\frac{\partial v}{\partial\theta}}{\sqrt{(r')^2+r^2}}\end{aligned}$$

2.) First, a sufficient condition for the existence of a solution is $\int_{\partial\Omega}g(s)ds=0$. Assume now that this condition is satisfied.

From the representation theorems we have

$$u(x)=\frac{1}{4\pi}\iint_{\partial\Omega}[\frac{1}{|y-x|}\frac{\partial u(y)}{\partial n}-u(y)\frac{\partial}{\partial n}\frac{1}{|y-x|}]d\sigma_y.$$

We also have $\iint_{\partial\Omega}[-u(y)\frac{\partial v(y)}{\partial n}+v(y)\frac{\partial u(y)}{\partial n}]d\sigma_y=0$, for any two harmonic functions u,v. Adding this expression to the previous one, we get that

$$u(x)=\iint_{\partial\Omega}\left([\frac{1}{4\pi|y-x|}+v(y)]\frac{\partial u(y)}{\partial n}-u(y)\frac{\partial}{\partial n}[\frac{1}{4\pi|y-x|}+v(y)]\right)d\sigma_y.$$

Lastly, we set $G(x,y)=\frac{1}{4\pi|y-x|}+v(y)$ and then we choose an harmonic function v that satisfy $v(y)=-\frac{1}{4\pi|y-x|}+c$ on $\partial\Omega$, for some $c\in\mathbb{R}$. Finally, $u(x)=\iint_{\partial\Omega}G(x,y)g(x)d\sigma_y$, $x\in\Omega$.

Problem 5:

1.) a.)(2pts) Prove that $\nabla f(u) = f'(u)\nabla u$.

b.)(4pts) Prove that $\Delta f(u) = f'(u)\Delta u + f''(u)|\nabla u|^2$.

c.)(6pts) Consider a function u that satisfy $u = \Delta u = 0$ for all $x \in \partial\Omega$, where Ω is a bounded domain of $\mathbb{R}^3$ and $\partial\Omega$ its smooth boundary. Find sufficient conditions on f such that there hold

$$\int_\Omega u\Delta f(u)dx = \int_\Omega f(u)\Delta u dx$$

and

$$\int_\Omega \Delta u\Delta^2 f(u)dx = \int_\Omega \Delta f(u)\Delta^2 u dx$$

2.)(8pts) Consider the DE

$$\frac{dy^2}{dt} + 2y^2 \leq f(t)y^2.$$

Assume $\int_0^t f(s)ds \leq t+2,\ \forall t \geq 0$. Deduce an estimate of $y^2(t)$ for $t \geq 0$.

Solution:

1.)a.) $\frac{\partial}{\partial x}(f(u)) = f'(u)\frac{\partial u}{\partial x}$, $\frac{\partial}{\partial y}(f(u)) = f'(u)\frac{\partial u}{\partial y} \Longrightarrow \nabla f(u) = \frac{\partial}{\partial x}(f(u))i + \frac{\partial}{\partial y}(f(u))j$
$= f'(u)\nabla u.$

b.) $\frac{\partial^2}{\partial x^2}(f(u)) = \frac{\partial}{\partial x}[f'(u)\frac{\partial u}{\partial x}] = f''(u)(\frac{\partial u}{\partial x})^2 + f'(u)\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}(f(u)) = \frac{\partial}{\partial y}[f'(u)\frac{\partial u}{\partial y}] = f''(u)(\frac{\partial u}{\partial y})^2 + f'(u)\frac{\partial^2 u}{\partial y^2} \Longrightarrow \Delta f(u) = f''(u)|\nabla u|^2 + f'(u)\Delta u.$

c.)

$$\left.\begin{aligned} I &= \int_\Omega u\Delta f(u)dx = -\int_\Omega \nabla u.\nabla f(u)dx + \underbrace{\int_{\partial\Omega} u\frac{\partial f(u)}{\partial n}ds}_{=0} \\ J &= \int_\Omega f(u)\Delta u dx = -\int_\Omega \nabla u.\nabla f(u)dx + \int_{\partial\Omega} f(u)\frac{\partial u}{\partial n}ds \end{aligned}\right\} I = J \Longleftrightarrow f(0) = 0$$

$$\left.\begin{aligned} K &= \int_\Omega \Delta u\Delta^2 f(u)dx = -\int_\Omega \nabla\Delta u.\nabla\Delta f(u)dx + \underbrace{\int_{\partial\Omega} \Delta u\frac{\partial\Delta f(u)}{\partial n}ds}_{=0} \\ L &= \int_\Omega \Delta f(u)\Delta^2 u dx = -\int_\Omega \nabla\Delta u.\nabla\Delta f(u)dx + \int_{\partial\Omega} \Delta f(u)\frac{\partial\Delta u}{\partial n}ds \end{aligned}\right\} I = J \Longleftrightarrow f''(0) = 0.$$

2.) We have $\frac{dy^2}{dt} + 2y^2 \leq f(t)y^2$ or

$$\frac{dy^2}{dt} + [2 - f(t)]y^2 \leq 0$$

We multiply by the integrating factor $e^{\int_0^t (2-f(s))ds} = e^{2t - \int_0^t f(s)ds}$, to find

$$\frac{d}{dt}[y^2 e^{2t-\int_0^t f(s)ds}] \leq 0$$

Lastly, we integrate from 0 to t, and we find

$$y^2 e^{2t-\int_0^t f(s)ds} - y^2(0) \leq 0,$$

that is,

$$y^2(t) \leq y^2(0)e^{-2t+\int_0^t f(s)ds} \leq y^2(0)e^{-2t+t+2} = y^2(0)e^{-t+2},\ \forall t \geq 0.$$