King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 568 Midterm Exam
The Second Semester of 2022-2023 (222)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1:

1.)(10pts)Find the solution of the quasi-linear equation

$$x^2 e^{\frac{1}{x}} u_x + u_y = u^2 - 1, \quad x > 0,\ y > 0$$

that passes through the curve $\Gamma : (s, -s, 0)$.

2.)(10pts) Find the canonical form of the second order PDE: $u_{xx} + 2u_{xy} + u_{yy} = y$.

Solution

1.) The characteristic equations are

$$\frac{dx}{dt} = x^2 e^{\frac{1}{x}} \Rightarrow \int x^{-2} e^{-\frac{1}{x}} dx = \int dt \Rightarrow e^{-\frac{1}{x}} = t + c_1$$

$$\frac{dy}{dt} = 1 \Rightarrow y = t + c_2$$

$$\frac{du}{dt} = u^2 - 1 \Rightarrow \int \frac{du}{u^2-1} = dt, \quad \frac{u-1}{u+1} = c_3 e^{2t},$$

At $t=0$, $\left.\begin{array}{l} e^{-\frac{1}{s}} = c_1 \\ -s = c_2 \\ -1 = c_3 \end{array}\right\} \Rightarrow$ $e^{-\frac{1}{x}} = t + e^{-\frac{1}{s}}$

$y = t - s \Rightarrow s = t - y$

$\frac{u-1}{u+1} = -e^{2t} \Rightarrow t = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|$

$$\Rightarrow \boxed{e^{-\frac{1}{x}} = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + e^{-\frac{1}{\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| - y}}}, \quad u \neq 1, -1$$

2.) $u_{xx} + 2u_{xy} + u_{yy} = y$. This is an elliptic PDE

A characteristic equation is $\frac{dy}{dx} = 1 \Rightarrow y - x = c$

$\xi = y - x$. Let $\eta = x \Rightarrow J = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$u_x = w_\xi \xi_x + w_\eta \eta_x = -w_\xi + w_\eta$

$u_{xx} = (-w_\xi + w_\eta)_\xi \xi_x + (-w_\xi + w_\eta)_\eta \eta_x = w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}$

$u_{xy} = (-w_\xi + w_\eta)_\xi \xi_y + (-w_\xi + w_\eta)_\eta \eta_y = -w_{\xi\xi} + w_{\xi\eta}$

$u_y = w_\xi \xi_y + w_\eta \eta_y = w_\xi$; $u_{yy} = w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y = w_{\xi\xi}$

$\Rightarrow \quad w_{\eta\eta} = y$

$$\boxed{w_{\eta\eta} = \xi + \eta}$$

Problem 2:
Consider IVP

$$\begin{cases} u_{tt} - u_{xx} = 0, & x > 0,\ t > 0, \\ u(x,0) = \phi(x), & x > 0, \\ u_t(x,0) = \varphi(x), & x > 0, \\ u(0,0) = u_t(0,0) = 0. \end{cases}$$

1.)(10pts) Show that the solution $u(x,t)$ of the system must satisfy $u(0,t) = 0$, $\forall t > 0$.
2.)(10pts) Given that the d'Alembert solution for a wave IVP is

$$v(x,t) = \int_{x-2t}^{x+2t} f(x)dx, \quad -\infty < x < \infty,\ t > 0,$$

write down $v(x,1)$ explicitly when

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Solution

1.) Let $\tilde{\phi}(x) = \begin{cases} \phi(x), & x > 0 \\ 0, & x = 0 \\ -\phi(-x), & x < 0 \end{cases}$ and $\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x > 0 \\ 0, & x = 0 \\ -\varphi(-x), & x < 0 \end{cases}$

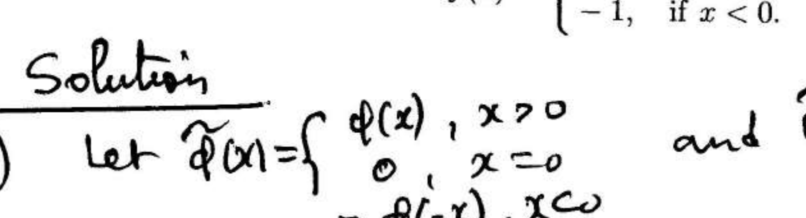
Consider the problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = \tilde{\phi}(x) \\ u_t(x,0) = \tilde{\varphi}(x) \end{cases} \Rightarrow u(x,t) = \frac{1}{2}\left[\tilde{\phi}(x-t) + \tilde{\phi}(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{\varphi}(s)\,ds$$

Notice that $\tilde{\phi}(-x) = -\tilde{\phi}(x)$, $\tilde{\varphi}(-x) = -\tilde{\varphi}(x)$

$\Rightarrow u(-x,t) = -u(x,t) \Rightarrow 2u(0,t) = 0$, $u(0,t) = 0$

2.)



$R_1: \{ x+2t < 0 \}$
$R_2: \{ x+2t > 0 \quad x-2t < 0 \}$
$R_3: \{ x-2t > 0 \}$

On R_1: $v(x,1) = -\int_{x-2}^{x+2} ds$; On R_2 $v(x,1) = -\int_{x-2}^{0} ds + \int_{0}^{x+2} ds$

On R_3: $v(x,1) = \int_{x-2}^{x+2} ds$

$$\Rightarrow \boxed{v(x,1) = \begin{cases} -4, & x < -2 \\ 2x, & x \in [-2,2] \\ 4, & x > 2 \end{cases}}$$

Problem 3:

1.)(14 pts) Solve the initial and boundary value problem

$$\begin{cases} v_{tt} - 9v_{xx} = 0, & 0 < x < \pi,\ t > 0 \\ v(x,0) = 0, & 0 \le x \le \pi, \\ v_t(x,0) = 1, & 0 \le x \le \pi, \\ v_x(0,t) = 0, & t \ge 0, \\ v(\pi,t) = 0, & t \ge 0. \end{cases} \tag{1}$$

2.)(6 pts) Solve the nonhomogeneous problem

$$\begin{cases} u_{tt} - u_{xx} = t & -\infty < x < \infty,\ t > 0, \\ u(x,0) = 1, & -\infty < x < \infty, \\ u_t(x,0) = 0, & -\infty < x < \infty. \end{cases}$$

Solution

1.) We use the method of separation of variables $V = XT$

$$XT'' = 9X''T \iff \frac{T''}{9T} = \frac{X''}{X} = \lambda \iff \begin{cases} X'' - \lambda X = 0 \\ X(\pi) = X'(0) = 0 \end{cases} \ \& \begin{cases} T'' - 9\lambda T = 0 \\ T(0) = 0 \end{cases}$$

• $\lambda = \alpha^2, \alpha > 0$, $m^2 - \alpha^2 = 0$, $m = \pm\alpha$, $X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$, $X' = c_1\alpha e^{\alpha x} - c_2\alpha e^{-\alpha x}$ $\begin{cases} c_1 e^{\pi\alpha} + c_2 e^{-\pi\alpha} = 0 \\ c_1 - c_2 = \end{cases} \Rightarrow c_1 = c_2 = 0$

• $\lambda = 0$, $m = 0,0$, $X = c_1 x + b$, $X' = c_1$, $\begin{cases} c_1\pi + b = 0 \\ c_1 = 0 \end{cases} \Rightarrow c_1 = b = 0$

• $\lambda = -\alpha^2$, $m = \pm i\alpha$, $X = c_1\cos\alpha x + c_2 \sin\alpha x$, $X' = -c_1\alpha\sin\alpha x + c_2\alpha\cos\alpha x$ $\begin{cases} c_1\cos\alpha\pi + c_2\sin\alpha\pi = 0 \\ c_2 = 0 \end{cases} \Rightarrow \cos\alpha\pi = 0$

$\alpha\pi = \frac{\pi}{2} + n\pi$

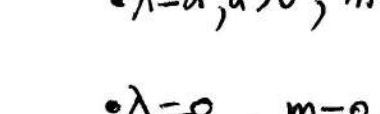
$X_n = C_n\cos(\tfrac{1}{2}+n)x$ $\quad T_n = c_1\cos(3(\tfrac{1}{2}+n)t) + c_2\sin 3(\tfrac{1}{2}+n)t$. $\quad T_n(0) = 0 \Rightarrow c_1 = 0$

$$\Rightarrow \boxed{v(x,t) = \sum_{n=0}^{\infty} A_n \cos(\tfrac{1}{2}+n)x \sin(3(\tfrac{1}{2}+n)t)}$$

$$1 = \sum_{n=1}^{\infty} A_n 3(\tfrac{1}{2}+n)\cos(\tfrac{1}{2}+n)x \Rightarrow 3(\tfrac{1}{2}+n)A_n = \frac{\int_0^\pi \cos(\frac{1}{2}+n)x\,dx}{\int_0^\pi \cos^2(\frac{1}{2}+n)x\,dx} = \frac{2(-1)^n}{(\frac{1}{2}+n)\pi}$$

$$\boxed{A_n = \frac{2(-1)^n}{3(\frac{1}{2}+n)^2\pi}}$$

2.)



$$u(x,t) = 1 + \frac{1}{2}\int_0^t \int_{T+x-t}^{-T+x+t} T\,dX\,dT = 1 + \int_0^t T(t-T)\,dT = \frac{t^3}{6} + 1$$

$$\boxed{u(x,t) = \frac{t^3}{6} + 1}$$

Problem 4:
Consider the Cauchy problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty, \\ u_t(x,0) = 0, & -\infty < x < \infty, \\ u(0,t) = 0, & t \geq 0. \end{cases} \tag{2}$$

1.)(16pts)Use the Fourier integral method to find the bounded solution of the problem.
2.)(4pts)Give an example of an unbounded solution of the equation such that $u(0,t) = u_t(0,t) = 0$.

Solution

1.) We use the separation of variables method $u = XT$

$XT'' = X''T \iff \frac{T''}{T} = \frac{X''}{X} = \lambda \iff \begin{cases} X'' - \lambda X = 0 \\ X(0) = 0 \end{cases}, \begin{cases} T'' - \lambda T = 0 \\ T'(0) = 0 \end{cases}$

- $\lambda = \omega^2, \omega > 0$, $m^2 - \omega^2 = 0$, $m = \pm\omega$, $X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$

$X(0) = 0 \Rightarrow c_2 = -c_1$, $X = c_1(e^{\omega x} - e^{-\omega x})$

$\lim_{x \to \pm\infty} X(x) = \pm\infty \Rightarrow c_1 = 0$ to have bounded solutions

- $\lambda = 0$, $m^2 = 0$, $m = 0, 0$, $X(x) = c_1 x + c_2$.

$X(0) = 0 \Rightarrow c_2 = 0$ But $\lim_{x \to \pm\infty} X(x) = \pm\infty \Rightarrow c_1 = 0$

- $\lambda = -\omega^2, \omega > 0$, $X(x) = c_1 \cos\omega x + c_2 \sin\omega x$

$X(0) = 0 \Rightarrow c_1 = 0 \qquad \Rightarrow X_\omega(x) = c_\omega \sin\omega x$

$T'' + \omega^2 T = 0 \Rightarrow T(t) = c_1 \cos\omega t + c_2 \sin\omega t$

$T' = -c_1 \omega \sin\omega t + c_2 \omega \cos\omega t$. $T'(0) = 0 \Rightarrow c_2 = 0$

$\Rightarrow \boxed{u(x,t) = \int_0^\infty A_\omega \sin\omega x \cos\omega t \, d\omega}$

$u(x,0) = f(x) = \int_0^\infty A_\omega \sin\omega x \, dx \Rightarrow A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \sin\omega x \, dx$

2.) We can see that $u(x,t) = x$ (from part 1) is an unbounded solution such that $u(0,t) = u_t(x,t) = 0$

Problem 5:
Consider the problem

$$\begin{cases} u_{tt} = u_{xx} + u_{yy}, & -1 < x < 1,\ 0 < y < 3,\ t > 0, \\ u(-1,y,t) = u(1,y,t), & 0 \le y \le 3,\ t > 0, \\ u_x(-1,y,t) = u_x(1,y,t), & 0 \le y \le 3,\ t > 0, \\ u_y(x,0,t) = u(x,3,t) = 0, & -1 \le x \le 1,\ t > 0, \\ u(x,y,0) = 0, & -1 \le x \le 1,\ 0 \le y \le 3, \\ u_t(x,y,0) = 2, & t \ge 0. \end{cases}$$

1.)(10pts) Using the separation of variables method, write down the independent systems satisfies by the three variables x, y and t. (Do not solve them).
2.)(10pts) Let the function

$$v(x,y,t) = u(t,x,y) + f(-t,2x).$$

Covert the first equation only into an equation satisfied by v (Do not solve the equation).

Solution

1.) $u = XYT \Rightarrow XYT'' = X''YT + XY''T \Leftrightarrow \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$

$\Rightarrow \frac{T''}{T} - \frac{Y''}{Y} = \frac{X''}{X} = \lambda \Leftrightarrow X'' - \lambda X = 0$

and

$\frac{T''}{T} - \frac{Y''}{Y} = \lambda \Leftrightarrow \frac{T''}{T} - \lambda = \frac{Y''}{Y} = \mu$

$\Rightarrow Y''$, $T'' - (\lambda + \mu)T = 0$

Thus, $$\boxed{\begin{matrix} X'' - \lambda X = 0 \\ X(-1) = X(1) \\ X'(-1) = X'(1) \end{matrix} \;;\; \begin{matrix} Y'' - \mu Y = 0 \\ Y'(0) = Y(3) = 0 \end{matrix} \;;\; \begin{matrix} T'' - (\lambda + \mu)T \\ T(0) = 0 \end{matrix}}$$

2.) $V(x,y,t) = u(x,y,t) + f(-t,2x)$

$V_t = u_t - \frac{\partial f}{\partial t}(-t,2x)$; $\quad V_{tt} = u_{tt} + \frac{\partial^2 f}{\partial t^2}(-t,2x)$

$V_x = u_x + 2\frac{\partial f}{\partial x}(-t,2x)$, $\quad V_{xx} = u_{xx} + 4\frac{\partial^2 f}{\partial x^2}(-t,2x)$

$V_{yy} = u_{yy}$

$\Rightarrow V_{tt} - \frac{\partial^2 f}{\partial t^2} = V_{xx} - 4\frac{\partial^2 f}{\partial x^2} + V_{yy}$

$$\boxed{V_{tt} = V_{xx} + V_{yy} + \frac{\partial^2 f}{\partial t^2}(-t,2x) - 4\frac{\partial^2 f}{\partial x^2}(-t,2x)}$$