MIDTERM MATH 568 T242

Problem 1:
Find a solution of the equation $y^2u_x + x^2u_y = y^2$ which satisfies
(a) $u(x, y) = x$ on $y = 4x$
(b) $u(x, y) = -2y$ on $y^3 = x^3 - 2$
(c) $u(x, y) = y^2$ on $y = -x$

Problem 2:
Find the explicit solution of the quasilinear equation $(x+u)u_x + (y+u)u_y = 2u$ which passes through the curve $(\Gamma) : x(s) = 1 - s,\ y(s) = 1 + s,\ u(s) = s$.

Problem 3:
Assume that u satisfies the inhomogeneous 1-D equation

$$u_{tt} = c^2u_{xx} + f(x, t),\ 0 < x < L,\ t > 0$$

Show that (a) the following integral relationship holds for any interval $[a, b]$ with $0 < a, b < L$:

$$\frac{1}{2}\frac{d}{dt}\int_a^b (u_t^2 + c^2u_x^2)dx = c^2u_tu_x\big|_a^b + \int_a^b fu_t dx.$$

(b) Show that the equation with $u(0, t) = u(L, t) = 0,\ u(x, 0) = g(x)$, for $0 \le x \le L,\ t \ge 0$ can have only one solution (Hint: Assume that there are two solutions and use the result of part (a) for the difference of these solutions).

Problem 4:
Consider the equation $xu_{xx} - yu_{yy} + \frac{1}{2}(u_x - u_y) = 0$. Find
(a) the domain where the equation is elliptic and where it is hyperbolic.
(b) For each of these domains, find the canonical transformation.

Problem 5:
Consider the equation $u_{xx} - 6u_{xy} + 9u_{yy} = xy^2$. Find
(a) a coordinates system (s, t) in which the equation has the form $9v_{tt} = \frac{1}{3}(s - t)t^2$
(b) the general solution $u(x, y)$
(c) a solution which satisfies $u(x, 0) = \sin x,\ u_y(x, 0) = \cos x$ for all $x \in R$.

Problem 6:
Consider the heat equation $u_t = \Delta u$ in a 2-D region Ω. Define the mass M by $M(t) = \int_\Omega u(x, t)dx$. Show that M is actually constant in time if the boundary condition is $\frac{\partial u}{\partial n} = 0$ ($\frac{\partial u}{\partial n}$ is the normal derivative of u).