King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 569 Final Exam
The First Semester of 2023-2024 (231)
Time Allowed: 180min

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1: Consider the scalar product

$$(\varphi, \psi) = \int_0^\pi \varphi(x)\psi(x)dx \quad \text{and} \quad |\varphi|^2 = (\varphi, \varphi)_1.$$

Consider the functions

$$e_n(x) = \cos(nx) \quad \text{and} \quad e_m(x) = \cos(mx), \quad n, m \in \mathbb{N},$$

1.)a.)(5pts) Show that $(e_0, e_m) = 0$ and $(e_n, e_m) = 0$, $\forall n \neq m$.
b.)(5pts) Compute $|e_0|^2$ and $|e_n|^2$, $\forall n \neq 0$.
2.)(10pts)Use the functions $e_n(x)$ and the spectral properties of the Laplace operator to deduce from part 1. an orthnormal and complete basis of $L^2(0, \pi)$. Justify your answer clearly.

Hint: You may need the relation $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$.

Solution:
1.)a.)• Assume $n \neq 0$. We have

$$(e_0, e_m) = \int_0^\pi \cos mx dx = \left[\frac{\sin mx}{m}\right]_0^\pi = 0.$$

• Assume $m \neq 0$, $m \neq 0$ and $n \neq m$. We have

$$\begin{aligned}(e_n, e_m) &= \int_0^\pi \cos nx \cos mx dx = \frac{1}{2}\int_0^\pi [\cos(n-m)x + \cos(n+m)x]dx \\ &= \frac{1}{2}\left[\frac{\sin(n-m)x}{n-m} + \frac{\sin(n+m)x}{n+m}\right]_0^\pi = 0.\end{aligned}$$

b.)• We have

$$|e_0|^2 = \int_0^\pi e_0^2(x)dx = \int_0^\pi dx = \pi$$

• Assume $n \neq 0$

$$\begin{aligned}|e_n(x)|^2 &= \int_0^\pi e_n^2(x)dx = \int_0^\pi \cos^2(nx)dx = \frac{1}{2}\int_0^\pi [1 + \cos(2nx)]dx \\ &= \frac{1}{2}\left[x + \frac{\sin(2nx)}{2n}\right]_0^\pi = \frac{\pi}{2}.\end{aligned}$$

2.) We define the Laplace operator

$$A = -\frac{d^2}{dx^2} : V = \{\varphi \in H^2(0, \pi), \varphi'(0) = \varphi'(\pi) = 0\} \to L^2(0, 1).$$

This operator $A : \dot{V} \to \dot{L}^2(0, 1)$ is self-adjoint, strictly positive with compact inverse. There exists a complete orthormal basis $\{w_0, w_n\}_{n=1,2,..}$ of $L^2(0, 1)$, made of eigenfunctions of A, that is,

$$Aw_n(x) = \lambda_n w_n(x).$$

We finally set $w_0(x) = \frac{1}{\sqrt{\pi}}$ and $w_n(x) = \sqrt{\frac{2}{\pi}}e_n(x)$.

Problem 2: Let Ω be an open bounded domain of $\mathbb{R}^3$ of class $\mathcal{C}^2$.
1.)(6pts) Consider the Poisson problem

$$\begin{cases} -\Delta u + 2u = f, \\ \dfrac{\partial u}{\partial n}|_{\partial\Omega} = 0. \end{cases} \tag{1}$$

What do you know about the existence and uniqueness of a weak solution u of (1)?. What is the regularity of u $f \in L^2(\Omega)$?
2.) Consider the Poisson problem

$$\begin{cases} -\Delta u = f, \\ \dfrac{\partial u}{\partial n}|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

a.)(3pts) Integrate (2) over Ω and deduce a necessary condition on f for the existence of a solution u to (2).
b.)(5pts) What do you know about the existence, uniqueness and the regularity of a weak solution u of (2) if $f \in L^2(\Omega)$.?
3.)(6pts) Consider the Poisson problem

$$\begin{cases} -\Delta u = f, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{3}$$

What do you know about the existence and uniqueness of a weak solution u of (3)?. What is the regularity of u $f \in L^2(\Omega)$?

Solution:
1.) For any $f \in (H^1(\Omega))'$, there exists a unique weak solution $u \in H^1(\Omega)$ to the Poisson problem (1). If $f \in L^2(\Omega)$, then $u \in \{\varphi \in H^2(\Omega),\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}$.

2.)a.) A necessary condition for the existence of a solution is $\int_\Omega f(x)dx = 0$. To see this, we integrate (2) over Ω, and we find

$$\underbrace{-\int_\Omega \Delta u dx}_{=\int_\Omega \nabla u, \nabla 1 dx = 0} = \int_\Omega f(x)dx$$

b.) For any $f \in (\dot{H}^1(\Omega))'$, there exists a unique weak solution $u \in \dot{H}^1(\Omega)$. If $f \in \dot{L}^2(\Omega)$, then $u \in \{\varphi \in \dot{H}^2(\Omega),\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}$.

3.) For any $f \in H^{-1}(\Omega)$, there exists a unique weak solution $u \in H_0^1(\Omega)$ of the Poisson problem (3). If $f \in L^2(\Omega)$, then $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Problem 3: Let Ω be an open bounded domain of $\mathbb{R}^3$ and consider the following initial and boundary value problem:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} - \Delta\rho + g(\rho) = f, \\ \rho|_{t=0} = \rho_0, \\ \rho|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

We assume that the nonlinear function $g \in \mathcal{C}^1(\mathbb{R})$ satisfies the condition:

$$g(0) = 0, \;\; \|\nabla g(\rho)\| \leq \|\Delta\rho\|, \qquad \text{where } \|\varphi\|^2 = \int_\Omega \varphi^2(x)dx. \tag{5}$$

We assume that $f \in L^2(0, T, L^2(\Omega))$, for any $T > 0$.
<u>We admit that</u>, if $\rho_0 \in H_0^1(\Omega)$, then Problem (4) has an approximate local solution $\rho_m(t)$, $\forall t \in [0, T_m)$, for some $T_m > 0$ [**Do not prove the existence of $\boldsymbol{\rho_m}$**] .

1.(10 pts) Show that $\rho_m \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, for any $T > 0$.
2.(10 pts) **Explain** how you can pass to the limit $m \to \infty$ in the term $g(\rho_m)$, given that $\|\frac{d\rho_m}{dt}\|_{L^2(0,T;L^2(\Omega))} \leq C$ (uniformly in m).

<u>**Solution:**</u>
1. We have

$$\frac{\partial \rho_m}{\partial t} - \Delta\rho_m + P_m g(\rho_m) = P_m f. \tag{6}$$

We multiply (6) by $-\Delta\rho_m$, integrate over Ω and we deduce

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho_m\|^2 + \|\Delta\rho_m\|^2 + \int_\Omega \nabla g(\rho_m).\nabla\rho_m dx - \underbrace{\int_{\partial\Omega} g(\phi_m)\frac{\partial}{\partial n}\frac{\partial \rho_m}{\partial t}dx}_{=0, \text{ since } g(0)=0} = -\int_\Omega P_m f \Delta\rho_m dx = 0. \tag{7}$$

$$|\int_\Omega \nabla g(\rho_m).\nabla\rho_m dx| \leq \|\nabla g(\rho_m)\|\|\nabla\rho_m\| \leq \|\Delta\rho_m\|\|\|\nabla\rho_m\| \leq \frac{1}{4}\|\Delta\rho_m\|^2 + \|\nabla\rho_m\|^2$$
$$|\int_\Omega P_m f \Delta\rho_m dx| \leq \|f\|\|\Delta\rho_m\| \leq \frac{1}{4}\|\Delta\rho_m\|^2 + \|f\|^2.$$

We deduce from (7) that

$$\frac{d}{dt}\|\nabla\rho_m\|^2 + \|\Delta\rho_m\|^2 \leq 2\|\nabla\rho_m\|^2 + 2\|f\|^2. \tag{8}$$

We can derive from that, if $\|\rho_0\| \leq R$, then

$$\|\rho_m\|_{L^\infty(0,T;H_0^1(\Omega))} = \sup_{t\in[0,T]} \|\nabla\rho_m(t)\| \leq C(R, T), \tag{9}$$

$$\|\rho_m\|^2_{L^2(0,T;H^2\cap H_0^1)} = \int_0^T \|\Delta\rho_m\|^2 dt \leq C(R, T), . \tag{10}$$

2. We deduce from (5) and (10) that $\int_0^T \|g(\rho_m)\|^2_{H_0^1} dt \leq C(R, T)$, and then $\frac{d\rho_m}{dt} \in L^2(0, T; L^2)$ (deduced from (4)). There exists a subsequence $\{\rho_m\}_m$ such that $\rho_m \to \rho$ strongly in $L^2(0, T; H_0^1(\Omega))$ and $\rho_m(x, t) \to \rho(x, t)$ a.e. in $\Omega \times (0, T)$. As g is continuous, it follows that

$$P_m g(\rho_m) \rightharpoonup g(\rho) \;\text{ in }\; L^2(0, T, H_0^1(\Omega)) \;\text{ weakly.}$$

Problem 4: Let Ω be an open bounded domain of $\mathbb{R}^2$, and consider the following initial and boundary value problem:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} - \Delta\rho + g(\rho) = 0, \\ \dfrac{\partial \rho}{\partial n}|_{\partial\Omega} = 0, \\ \rho|_{t=0} = \rho_0. \end{cases} \tag{11}$$

We assume that the function $g \in \mathcal{C}^1(\mathbb{R})$ satisfies the conditions:

$$|g'(u)| \leq 1 + |u|^p, \quad \text{and} \quad \int_\Omega |u|^{4p} dx \leq 1, \quad \text{for some } p > 0. \tag{12}$$

Let

$$H = \{\varphi \in H^1(\Omega), \quad \int_\Omega \varphi(x) dx = 0\},$$

$$V = \left\{\varphi \in H^2(\Omega), \ \frac{\partial \rho}{\partial n}|_{\partial\Omega} = 0, \quad \int_\Omega \varphi(x) dx = 0\right\}.$$

1.(10 pts) We assume that $\rho_0 \in V$. Show that $\rho \in L^\infty(0,T;V), \quad \forall T > 0$.
Hint: You may multiply the equation by $-\Delta\frac{\partial u}{\partial t}$.
2.a.)(6 pts) Show that $\frac{d\rho}{dt} \in L^2(0,T;H)$.
b.) (4 pts)Assume $u \in L^2(0,T;H^3 \cap V)$. What can you state about the continuity of the map $t \to u(t)$?

Solution:

1. We multiply the problem by $-\Delta\frac{\partial \rho}{\partial t}$, integrate over Ω and we find

$$\frac{1}{2}\frac{d}{dt}\|\Delta\rho\|^2 + \|\nabla\frac{\partial\rho}{\partial t}\|^2 + \int_\Omega g'(\rho)\nabla\rho.\nabla\frac{\partial\rho}{\partial t}dx - \underbrace{\int_{\partial\Omega} g(\phi_m)\frac{\partial}{\partial n}\frac{\partial\rho_m}{\partial t}dx}_{=0, \text{ since } \frac{\partial}{\partial n}\frac{\partial\rho_m}{\partial t}=0} = 0, \tag{13}$$

But, $\|g'(\rho)\|_{L^4}^4 \leq c\int_\Omega(1+|\rho|^{4p})dx \leq C \Longrightarrow \|g'(\rho)\|_{L^4} \leq C$, and

$$\int_\Omega |g'(\rho)||\nabla\rho||\nabla\frac{\partial\rho}{\partial t}|dx \leq \|g'(\rho)\|_{L^4}\|\nabla\rho\|_{L^4}\|\nabla\frac{\partial\rho}{\partial t}\|_{L^2}$$

$$\leq \frac{1}{2}\|\nabla\frac{\partial\rho}{\partial t}\|^2 + c\|\Delta\rho\|^2, \tag{14}$$

We deduce from (13) and (14) that

$$\frac{d}{dt}\|\Delta\rho\|^2 + \|\nabla\frac{\partial\rho}{\partial t}\|^2 \leq c\|\Delta\rho\|^2. \tag{15}$$

If $\|\Delta\rho_0\| \leq R$, the we can deduce from that

$$\frac{d}{dt}\|\Delta\rho\|^2 \leq c\|\Delta\rho\|^2 \Longrightarrow^{\text{by Gromwall inequality}} \|\rho\|_{L^\infty(0,T;V)} \leq C,$$

2.)a.) Integrate (15) $\Longrightarrow \|\Delta\rho(t)\|^2 + \int_0^t \|\nabla\frac{\partial\rho}{\partial t}\|^2 ds \leq \|\Delta\rho_0\|^2 + c\int_0^T \|\Delta\rho\|^2 ds \leq C(T)$

$$\Longrightarrow \|\frac{\partial\rho}{\partial t}\|_{L^2(0,T;H)} \leq C.$$

b.) $u \in L^2(0,T;H^3 \cap V)$ and $\frac{\partial\rho}{\partial t} \in L^2(0,T;H) \Longrightarrow u \in \mathcal{C}([0,T],V)$.

Problem 5: Let Ω be an open bounded set of $\mathbb{R}^3$, with smooth boundary $\partial\Omega$. We consider the initial and boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta \dfrac{\partial u}{\partial t} - \Delta u + g(u) = f, \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{16}$$

We assume that g satisfies the following condition

$$|g(s) - g(r)| \leq |s - r|(|s|^4 + |r|^4), \ \ \forall s, r \in \mathbb{R}, \tag{17}$$

We admit that, if $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, then the problem possesses at least one weak solution u and, for any $T > 0$, there is a constant $C(T)$ such that

$$\|\nabla u(t)\| + \|\frac{\partial u}{\partial t}(t)\| \leq C(T), \quad \forall t \in [0, T], \qquad \text{where} \quad \|\varphi\|^2 = \int_\Omega \varphi(x)^2 dx.$$

1.(5 pts) Let u_1 and u_2 be two solutions of Problem 16. Let $w = u_1 - u_2$.
Write the problem satisfied by w.
2.(15 pts) Multiply by $\frac{\partial w}{\partial t}$ and show that $\|\nabla w(t)\|^2 + \|\frac{\partial w}{\partial t}(t)\|^2 = 0$, $\forall t \in [0, T]$.

Solution:
1. We have

$$\begin{cases} w_{tt} - \Delta w_t - \Delta w + g(u_1) - g(u_2) = 0, \\ w|_{t=0} = 0, \quad w_t|_{t=0} = 0, \\ w|_{\partial\Omega} = 0, \end{cases} \tag{18}$$

2. We multiply the first equation of (18) by $\frac{dw}{dt}$, and we integrate over Ω, and we deduce

$$\frac{1}{2}\frac{d}{dt}[\|\frac{dw}{dt}\|^2 + \|\nabla w\|^2] + \|\nabla \frac{dw}{dt}\|^2 \leq \int_\Omega |g(u_1) - g(u_2)||\frac{dw}{dt}|dx,$$

We have

$$\begin{aligned} \int_\Omega |g(u_1) - g(u_2)||\frac{dw}{dt}|dx &\leq \int_\Omega \underbrace{(|u_1|^4 + |u_2|^4)}_{\in L^{\frac{3}{2}}} \underbrace{|w|}_{\in L^6} \underbrace{|\frac{dw}{dt}|}_{\in L^6} dx \\ &\leq c(\|u_1\|^4_{L^6(\Omega)} + \|u_2\|^4_{L^6(\Omega)})\|w\|_{L^6(\Omega)}\|\nabla \frac{dw}{dt}\| \\ &\leq c(\|\nabla u_1\|^4 + \|\nabla u_2\|^4)\|\nabla w\|\|\nabla \frac{dw}{dt}\| \\ &\leq \frac{1}{2}\|\nabla \frac{dw}{dt}\|^2 + (\|\nabla u_1\|^8 + \|\nabla u_2\|^8)\|\nabla w\|^2. \end{aligned}$$

Now, we notice that $\|\nabla u_1(t)\|^8 + \|\nabla u_2(t)\|^8 \leq C(T)$, $\forall t \in [0, T]$. Thus, we get that

$$\frac{d}{dt}\left(\|\frac{dw}{dt}\|^2 + \|\nabla w\|^2\right) \leq C\left(\|\frac{dw}{dt}\|^2 + \|\nabla w\|^2\right). \tag{19}$$

We apply the Gronwall's lemma and we find

$$\|\frac{dw}{dt}(t)\|^2 + \|\nabla w(t)\|^2 \leq e^{CT} \underbrace{\left(\|\frac{dw}{dt}(0)\|^2 + \|\nabla w(0)\|^2\right)}_{=0}, \quad \forall t \in [0, T].$$

$\Longrightarrow \|\frac{dw}{dt}(t)\|^2 + \|\nabla w(t)\|^2 = 0, \quad \forall t \in [0, T].$