King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 569 Midterm Exam
The First Semester of 2023-2024 (231)

Time Allowed: 130mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1: Let Ω be an open bounded domain of $\mathbb{R}^3$ of class $\mathcal{C}^3$.
1.)a.)(5pts) Explain how to obtain eigenfunctions $\{e_j(x)\}$ and corresponding eigenvalues $\{\lambda_j\}$ of the operator $A = -\Delta$ subject to Dirichlet boundary conditions.
b)(5pts) Is it possible that the first eigenvalue λ_1 of the operators A is equal to zero? Justify your answer.
2.)(10pts) Complete the dots in the definitions of the domain of the operator A and $A^{\frac{3}{2}}$

$$D(A) = \{\varphi(x) = \sum_{j=1}^{\infty} \alpha_j e_j(x) \quad \text{such that} \quad ....\}$$

$$D(A^{\frac{3}{2}}) = \{\varphi(x) = \sum_{j=1}^{\infty} \alpha_j e_j(x) \quad \text{such that} \quad ....\}$$

Solution:
1.) a.) Consider the operator

$$A = -\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$$

The operator A is self-adjoint, strictly positive with compact inverse A^{-1}. A theory spectral theorem shows that there exist a complete orthonormal basis $\{e_j\}_j$ of $L^2(\Omega)$ made of eigenfunctions of A and corresponding eigenvalues $\{\lambda_j\}$, $j = 1, 2, ...,$, such that

$$0 < \lambda_1 < \lambda_2 < .... < \lambda_k < ..... < ..... + \infty.$$

b.) The first eigenvalue λ_1 is strictly positive. Otherwise, if $\lambda_1 = 0$, it means that $Ae_1 = \lambda_1 e_1 = 0$, that is, $Ae_1 = 0$, or $e_1 = A^{-1}0 = 0$. But, we know that $e_1 = 0$ cannot be an eigenvector.

2.) We have $\|A\varphi\| = (A\varphi, A\varphi)$. Then

$$\begin{aligned} D(A) &= \{\varphi \in L^2, \ A\varphi \in L^2(\Omega)\} = \{\varphi \in L^2, \ \|A\varphi\| < \infty\} \\ &= \{\varphi(x) = \sum_{j=1}^{\infty} \alpha_j e_j(x) \quad \text{such that} \quad (\sum_{j=1}^{\infty} \alpha_j A e_j(x), \sum_{j=1}^{\infty} \alpha_j A e_j(x)) < \infty\} \\ &= \{\varphi(x) = \sum_{j=1}^{\infty} \alpha_j e_j(x) \quad \text{such that} \quad \sum_{j=1}^{\infty} \alpha_j^2 \lambda_j^2 < \infty\} \end{aligned}$$

We also have $\|A^{\frac{3}{2}}\varphi\| = (A^{\frac{3}{2}}\varphi, A^{\frac{3}{2}}\varphi)$. Then

$$\begin{aligned} D(A^{\frac{3}{2}}) &= \{\varphi \in L^2, \ A^{\frac{3}{2}}\varphi \in L^2(\Omega)\} = \{\varphi \in L^2, \ \|A^{\frac{3}{2}}\varphi\| < \infty\} \\ &= \{\varphi(x) = \sum_{j=1}^{\infty} \alpha_j e_j(x) \quad \text{such that} \quad (\sum_{j=1}^{\infty} \alpha_j A^{\frac{3}{2}} e_j(x), \sum_{j=1}^{\infty} \alpha_j A^{\frac{3}{2}} e_j(x)) < \infty\} \\ &= \{\varphi(x) = \sum_{j=1}^{\infty} \alpha_j e_j(x) \quad \text{such that} \quad \sum_{j=1}^{\infty} \alpha_j^2 \lambda_j^3 < \infty\} \end{aligned}$$

Problem 2: Let Ω be an open bounded domain of $\mathbb{R}^3$ of class $\mathcal{C}^2$. Consider the spaces

$$V = \{\varphi \in H^2(\Omega),\ \frac{\partial \varphi}{\partial n}|_{\partial\Omega} = 0\}$$
$$W = \{\varphi \in H^2(\Omega),\ \frac{\partial \varphi}{\partial n}|_{\partial\Omega} = 0,\ \int_\Omega \varphi(x)dx = 0\}$$
$$H == \{\varphi \in L^2(\Omega),\ \int_\Omega \varphi(x)dx = 0\}$$

1.) a.)(10pts) Explain how to obtain a basis of $L^2(\Omega)$ by considering the eigenvalue problem for the Laplace operator $N = -\Delta$ subject to Neumman boundary conditions.
b.)(5pts) Consider the operator

$$T = 2I + N^{\frac{1}{2}} + N : V \to L^2(\Omega).$$

Find the expression of ζ_j such that $Tw_j = \zeta_j w_j$, where $\{w_j\}$ are the eigenfunctions of the operator N.
2.)(5pts) Consider the linear operators

$$\begin{aligned} B &= -\Delta : V \to H, \\ C &= (-\Delta)^{\frac{1}{2}} : H^1(\Omega) \to H, \\ D &= I - \Delta : V \to L^2, \\ E &: (I-\Delta)^{-\frac{1}{2}} : (H^1(\Omega))' \to H^1(\Omega). \end{aligned}$$

Which ones of the above operators are isomorphism?

Solution:
Consider the operator
$$N = -\Delta : W \to H$$

The operator A is self-adjoint, strictly positive with compact inverse N^{-1}. A theory spectral theorem shows that there exist a complete orthonormal basis $\{w_j\}_j$ of H made of eigenfunctions of N and corresponding eigenvalues $\{\mu_j\}$, $j = 1, 2, ...,$, such that

$$0 < \mu_1 < \mu_2 < .... < \mu_k < ..... < ..... + \infty.$$

Now, we notice that if $w_0(x) = \frac{1}{\sqrt{|\Omega|}}$ then $w_) \in V$ and we have $Nw_0(x) = 0w_0(x)$, that is w_0 is an eigenvalue of N corresponding to the eigenvalue $\mu_0 = 0$. Thus the family $\{w_j\}$, $j = 0, 1, 2, , ,$ is a complet basis we orhtonormal basis of $L^2(\Omega)$.
2.) We have $N^{\frac{1}{2}}w_j = \sqrt{\lambda_j}w_j$ and $Nw_j = \lambda_j w_j$, hence

$$\begin{aligned} Tw_j &= (2I + N^{\frac{1}{2}} + N)w_j = 2w_j + N^{\frac{1}{2}}w_j + Nw_j \\ &= (2 + \sqrt{\lambda_j} + \lambda_j)w_j \end{aligned}$$

therefore

$$\zeta_j = 2 + \sqrt{\lambda_j} + \lambda_j, \quad j = 0, 1, 2, ...$$

2.)Only the operator D is an isomorphism.

Problem 3: Let Ω be an open **unit disk** of $\mathbb{R}^2$ and let $f \in L^2(\Omega)$.
Let consider the function $b : \Omega \to \mathbb{R}$ defined by

$$b(x,y) = \frac{1}{1+x^2+y^2}.$$

We now consider the BV problem

$$\begin{cases} 2\phi - \Delta\phi - b(x,y)\phi = f, & x \in \Omega, \\ \dfrac{\partial \phi}{\partial n}(x) = 0, & x \in \partial\Omega. \end{cases}$$

Use Lax-Milgram theorem to show that this BVP has a unique **weak solution**.

Solution:

$$a(\phi, q) = 2\int_\Omega \phi q dx + \int_\Omega \nabla\phi.\nabla q dx - \int_\Omega b(x,y)\phi q dx = \int_\Omega f q dx, \quad \forall q \in H^1(\Omega).$$

We have $0 \le x^2+y^2 \le 1 \Longrightarrow 1 \le 1+x^2+y^2 \le 2 \Longrightarrow \frac{1}{2} \le \frac{1}{1+x^2+y^2} \le 1$

$$\Longrightarrow \quad -1 \le -b(x,y) \le -\frac{1}{2}.$$

The bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is continuous and coercive.
Indeed, we have $H^1(\Omega) \subset L^2(\Omega)$, that is, $\|\varphi\| \le c\|\varphi\|_{H^1}$ and then

$$\begin{aligned} |a(\varphi, q)| &\le 2\int_\Omega |\varphi||q|dx + \int_\Omega |\nabla\varphi||\nabla q|dx + \int_\Omega b(x,y)|\varphi||q|dx \\ &\le 2\|\varphi\|\|q\| + \|\nabla\varphi\|\|\nabla q\| + +\|\varphi\|\|q\| \\ &\le c_0\|\varphi\|_{H^1}\|q\|_{H^1}, \quad \forall \varphi, q \in H^1. \end{aligned}$$

We also have

$$\begin{aligned} a(u,u) &= \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx - \int_\Omega b(x,y)u^2 dx \\ &\ge 2\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx - \int_\Omega u^2 dx \\ &\ge \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx. \end{aligned}$$

We finally get that

$$a(u,u) \ge \|u\|^2_{H^1}.$$

As $f \in L^2(\Omega)$, it implies that $f \in (H^1(\Omega))'$, we apply the Lax Milgram theorem to prove that there exists a unique function $u \in H^1(\Omega)$ that is the weak solution of the BVP.

Problem 4: Let Ω be an open bounded domain of $\mathbb{R}^3$ of class $\mathcal{C}^2$.
Given $f \in L^2(\Omega)$ and a value $\varepsilon \in [1,5]$, we consider the BV problem

$$\begin{cases} -\Delta\phi + \phi^{2\varepsilon-1} = f, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

1.)(3pts)Write a weak formulation for the problem.
2.)(7pts)Consider an approximate problem and show the existence of an approximate solution $\phi_m(x)$ of the BVP. **Be brief and concise, no need to give too much details**.
3.)(10pts) Obtain all useful estimates that are bounded by a constant independent of m and that are needed for the passage to the limit in approximate problem.
(**Do not prove the passage to the limit**).

Solution:

1.) $$\int_\Omega \nabla\phi.\nabla q dx + \int_\Omega \phi^{2\varepsilon-1} q dx = \int_\Omega f q dx, \quad q \in V = H_0^1(\Omega) \cap L^{2\varepsilon}(\Omega), \tag{1}$$

In space dimension one and two, we can take $V = H_0^1$. In space dimension three, we can take $V = H_0^1$, if $1 \le 2\varepsilon \le 6$.
2.) We know that there exist a complete orthonormal family of eigenfunctions $\{e_j\}$ in $H^2(\Omega) \cap H_0^1$, and even in H^s such that $H^s \subset L^{2\varepsilon}$ if $\Omega \in \mathcal{C}^s$, and corresponding eigenvalues $\{\lambda_j\}$ such that $-\Delta e_j(x) = \lambda_j e_j(x)$, where

$$0 < \lambda_1 < \lambda_2 < .... < \lambda_k < .... + \infty.$$

Let $m \in \mathbb{N}^*$ and $E_m = \text{span}\{e_1, ..., e_m\}$. We now look for $\phi_m(x) = \sum_{j=1}^m c_j e_j(x)$ that is solution of the following approximate problem

$$\int_\Omega \nabla\phi_m.\nabla q dx + \int_\Omega \phi_m^{2\varepsilon-1} q dx = \int_\Omega f q dx, \quad q \in E_m. \tag{2}$$

Taking $q = e_j$, for $j = 1, 2, ..., m$, we find that is equivalent to the vector equation

$$MY + F(Y) = 0,$$

where $Y = (c_1, c_2, ..., c_m)$ and $F(Y)$ a nonlinear function. This system has a unique solution Y by the Brauer fixed point theorem. Hence, the existence of the approximate solution $\phi_m \in E_m$.
3.) We take $q = \phi_m$ in (1) and we find

$$\|\phi_m\|_{H_0^1}^2 + \int_\Omega \phi_m^{2\varepsilon} dx = \int_\Omega f\phi_m dx \le |\int_\Omega f\phi_m dx| \le \|f\|_{-1}\|\phi_m\|_{H_0^1} \le \frac{1}{2}\|\phi_m\|_{H_0^1}^2 + \frac{1}{2}\|f\|_{-1}^2,$$

hence

$$\|\phi_m\|_{H_0^1}^2 + 2\int_\Omega |\phi_m|^{2\varepsilon} dx \le c_0^2\|f\|^2 = C \quad \Longrightarrow \quad .\|\phi_m\|_{H_0^1} \le C, \quad \|\phi_m\|_{L^{2\varepsilon}}^{2\varepsilon} \le \frac{1}{2}C.$$

where the constant C does is independent of m and where we used $\|f\|_{-1} \le c_0\|f\|$.
We also have

$$\|\phi_m^{2\varepsilon-1}\|_{L^{\frac{2\varepsilon}{2\varepsilon-1}}(\Omega)}^{\frac{2\varepsilon}{2\varepsilon-1}} = \int_\Omega |\phi_m^{2\varepsilon-1}|^{\frac{2\varepsilon}{2\varepsilon-1}} dx = \int_\Omega |\phi_m|^{2\varepsilon} dx = \|\phi_m\|_{L^{2\varepsilon}}^{2\varepsilon} \le \frac{1}{2}C.$$

Problem 5: Let Ω be an open bounded domain of $\mathbb{R}^3$ of class $\mathcal{C}^2$. Given $f \in L^2(\Omega)$, we consider the BV problem

$$\begin{cases} -\Delta\phi + \phi^5 = f, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

1.)(3pts) Write the weak variational formulation of the approximate problem.
2.)(12pts) Assume the approximate solutions $\phi_m(x)$ of the approximate BVP do exist, and satisfy the estimates

$$\|\nabla\phi_m\|^2 \leq C \quad \text{and} \quad \int_\Omega |\phi_m(x)|^6 dx < C,$$

where C is independent of m.
Explain how we can deduce a solution of the problem by passing to the limit $m \to \infty$ in the approximate problem. **Be brief, clear and precise.**
3.)(5pts) Prove that the solution of the BVP is unique.

Solution:

1.) $$\int_\Omega \nabla\phi_m.\nabla q dx + \int_\Omega \phi_m^5 q dx = \int_\Omega f q dx, \quad q \in E_m = \text{span}\{e_1, ..., e_m\}. \tag{3}$$

2.) • If $\|\phi_m\|_{H_0^1} \leq C$, then there exists a subsequence $\{\phi_m\}_m$ such that $\phi_m \rightharpoonup \phi$ in H_0^1, that is,

$$\int_\Omega \nabla\phi_m.\nabla q dx \to \int_\Omega \nabla\phi.\nabla q dx \quad \text{as } m \to \infty$$

• If $\|\phi_m\|_{L^6(\Omega)} \leq C$, then $\|\phi_m^5\|_{L^{\frac{6}{5}}}^{\frac{6}{5}} = \int_\Omega |\phi_m^5|^{\frac{6}{5}} dx = \int_\Omega \phi_m^6 dx \leq C$ and there exists a subsequence $\{\phi_m\}_m$ such that $\phi_m^5 \rightharpoonup \chi$ in $L^{\frac{6}{5}}$. Now, as the function $g(x) = x^5$ is continuous and $\phi_m \to \phi$ in L^2 and a.e. in Ω, it follows that $\phi_m^5 \to \phi^5$ a.e. in Ω. We apply a Lemma form the lecture notes that shows that $\phi_m^5 \rightharpoonup \phi^5$. that is, $chi = \phi^5$ and

$$\int_\Omega \phi_m^5 q dx \to \int_\Omega \phi^5 q dx \quad \text{as } m \to \infty.$$

With this, we can pass to the limit $m \to \infty$ in the approximate problem (3) to find

$$\int_\Omega \nabla\phi.\nabla q dx + \int_\Omega \phi^5 q dx = \int_\Omega f q dx, \quad q \in H_0^1(\Omega),$$

3.) Setting $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are two solutions of the BV problem, we have

$$\int_\Omega \nabla\phi.\nabla q dx + \int_\Omega (\phi_1^5 - \phi_2^5) q dx = 0, \quad q \in H_0^1(\Omega). \tag{4}$$

in particular, for $q = \phi$, we have

$$\|\phi\|_{H_0^1} + \int_\Omega (\phi_1^5 - \phi_2^5)\phi dx = 0. \tag{5}$$

But, the mean value theorem says $\phi_1^5 - \phi_2^5 = (\phi_1 - \phi_2)\int_0^1 (4(s\phi_1 + (1-s)\phi_2)^4 ds$. Thus, $\int_\Omega (\phi_1^5 - \phi_2^5)\phi dx = 4\int_\Omega \int_0^1 (s\phi_1 + (1-s)\phi_2)^4 \phi^4 ds dx \geq 0$, and we deduce that $\|\phi\|_{H_0^1} = 0$ and $\phi = 0$, hence $\phi_1 = \phi_2$ and the solution of the BVP is unique.