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Department of Mathematics, KFUPM

Math 572 (231): Numerical analysis of PDEs

Instructor: Khaled Furati

Final Exam

Duration: 180 minutes

Student Name: ……………………………………………

Q	1	2	3	4	5	Total	Bonus I	Bonus II
Max	13	13	13	15	16	70	5	5
Points								

Problem 1

Consider the problem

$$-u'' + 2u' + e^x u = x, \qquad 0 < x < 1,$$
$$u(0) = 2, \qquad u(1) = 3.$$

Let u_h be the continuous piecewise-linear finite element approximation on a uniform grid.

Define the global basis functions for the solution set S_h and the test space T_h.

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Problem 2

In the weak formulation of the wave equation $u_{tt} = u_{xx}$ subject to homogeneous Dirichlet BCs we find $u_h(t) \in S_h$ for each t such that

$$(d_{tt}u_h, \chi) + a(u_h, \chi) = (f, \chi), \qquad \forall \chi \in S_h, \qquad t > 0,$$
$$u_h(0) = v_h, \qquad d_t u_h(0) = w_h,$$

where $a(v, w) = (v', w')$.

Write the fully discrete scheme for $U^n \in S_h$ that approximates $u(t_n)$. Describe U^0 and U^1.

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Problem 3

Consider the following two-point boundary value problem

$$-u'' - 2u' = f(x), \qquad 0 < x < 1,$$
$$u(0) = 0, \qquad u'(1) + u(1) = 0,$$

where f is a given smooth function. Let v denote suitable test functions.

Derive the following weak formulation

$$(u' + 2u, v') = u(1)v(1) + (f, v), \quad \text{provided that } v(0) = 0.$$

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Problem 4

Consider the following two-point boundary value problem

$$-u'' = 1, \qquad 0 < x < 1,$$
$$u'(0) = u(1) = 0.$$

Partition the interval [0,1] using the uniform grid $\{x_0, x_1, x_2\} = \{0, 1/2, 1\}$ and let S_h be the space of piecewise-linear continuous functions on this partition vanishing at $x = 1$.

a) Determine the analytical expression for the hat basis functions ϕ_0 and ϕ_1.
b) Formulate the finite element method using S_h.
c) Write the numerical scheme in matrix form, then solve the linear system.

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Problem 5

Consider the following problem

$$u_t - u_{xx} = f(x), \qquad 0 < x < 1, \qquad t > 0,$$
$$u(t,0) = 0, \qquad u(t,1) = 0,$$
$$u(x,0) = v(x), \qquad 0 < x < 1.$$

The week formulation of this problem is to find $u = u(x,t) \in H_0^1$ such that

$$(u_t, \phi) + a(u, \phi) = (f, \phi), \qquad \forall \phi \in H_0^1, \qquad t > 0.$$
$$u(x,0) = v(x), \qquad x \in \Omega.$$

where $a(u,v) = \int_0^1 v'w'dx$. Consider a uniform mesh with step size $h = 1/M$ and a standard basis $\{\phi_j\}_{j=1}^{M-1}$ of the solution space S_h.

a) Define the semidiscrete finite element solution $u_h(x,t) = \sum_{j=1}^{M-1} \alpha_j(t)\, \phi_j(x)$ and write the system of ODEs for $\alpha(t) = \big(\alpha_1(t), \dots, \alpha_M(t)\big)$ in matrix form.
b) Derive the backward Euler-Galerkin scheme for $\alpha^n = \alpha(t_n)$.

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Bonus problem I

Explain what is meant by a priori and posterior error estimates. Which type is desirable for algorithms?

Bonus problem II

In Q5, derive the stability estimate

$$\|u_h(t)\| \le \|v_h\| + \int_0^t \|f\|\, ds.$$

Hint. Start with the weak formulation

$$(d_t u_h, \chi) + a(u_h, \chi) = (f, \chi), \qquad \forall \chi \in S_h, \qquad u_h(0) = v_h.$$

Then use $(w^2)' = 2ww'$ and Schwarz inequality $(w,z) \le \|w\|\,\|z\|$.

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