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Department of Mathematics, KFUPM

Math 572 (231): Numerical analysis of PDEs

Instructor: Khaled Furati

Midterm Exam

Duration: 110 minutes

Student Name: ……………………………………………

Q	1	2	3	4	Total
Max	16	9	15	12	52
Points					

1) True or false (Give a very brief justification)

   a) When the backward difference method with $h = 0.1$ is used to solve $u_t + 2u_x = 0$ then the possible largest time step for which the scheme satisfies the CFL condition is 0.05.

   b) If the approximate numerical solution U^n, $n \geq 1$, of the IVP
   $$u_t = u_{xx}, \qquad u(x,0) = v(x),$$
   is bounded by $\|v\|_\infty$, then the method is stable.

   c) For stable implementation of the explicit Euler method for $u_t = u_{xx}$ with $h = 0.1$, we can use $k = h/2$.

   d) Consider the equation $u'' - u' = 0$. Then the scheme
   $$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{u_{i+1} - u_i}{h} = 0$$
   is second order accurate.

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2) Consider the problem
$$u_t = 2u_x, \qquad x \in \mathbb{R}, \qquad t > 0,$$
$$u(\cdot, 0) = v, \qquad x \in \mathbb{R}.$$

   a) Construct the forward differencing (upwind) scheme.

   b) Show that if $\lambda \leq 1/2$, $\lambda = k/h$, then
   $$\|U^n\|_C \leq \|v\|_C.$$

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3) Consider the Dirichlet problem

$$-\Delta u = f, \qquad (x,y) \in \Omega = (0,1) \times (0,1),$$
$$u = 0, \qquad on\ \Gamma := \partial\Omega.$$

Introduce the grid points $(x_i, y_j) = (ih, jh),\ 0 \le i,j \le M.$

a) Construct the 5-point finite-difference scheme using the centered differencing for the second order derivative. Let U_{ij} denote the approximation of $u(x_i, y_j)$.

b) Show that if $f_{ij} \le 0$ in Ω, then the numerical solution attains its maximum on the boundary.

c) Formulate the matrix form of the scheme for $h = 1/5$.

d) Show that truncation error is

$$\tau_{ij} = \frac{h^2}{12}\left(\partial_x^4 u(\xi_i, y_j) + \partial_y^4 u(x_i, \eta_j)\right), \qquad (\xi_i, \eta_j) \in (x_{i-1}, x_{i+1}) \times (y_{j-1}, y_{j+1}).$$

e) Show that

$$\max_{ij}|U_{ij} - u_{ij}| \le Ch^2 \max_{\Omega}\{|\partial_x^4 u|, |\partial_y^4 u|\}.$$

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4) Consider the initial-boundary value problem

$$u_t = u_{xx} - u, \qquad 0 < x < 1, \qquad t > 0.$$
$$u(x,0) = x(1-x), \qquad 0 < x < 1,$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

Introduce the mesh-points $(x_j, t_n) = (jh, nk),\ j = 0,1, \dots M,\ n = 0, 1, 2, \dots$.

a) Construct an explicit finite difference scheme.

b) Write down the equations in matrix form.

c) Find the solution at first time step when $h = 1/4$ and $k = 1/32$.

d) Perform the von-Neumann stability analysis and show that the scheme is stable for

$$r < \frac{1}{2} - \frac{k}{4}, \qquad \text{where } r = \frac{k}{h^2}.$$