

Department of Mathematics, KFUPM
Math 572 (231): Numerical analysis of PDEs
 Instructor: Khaled Furati
 Midterm Exam
 Duration: 110 minutes

Student Name:

Q	1	2	3	4	Total
Max	16	9	15	12	52
Points					

1) True or false (Give a very brief justification)

a) When the backward difference method with $h = 0.1$ is used to solve $u_t + 2u_x = 0$ then the possible largest time step for which the scheme satisfies the CFL condition is 0.05.

b) If the approximate numerical solution U^n , $n \geq 1$, of the IVP

$$u_t = u_{xx}, \quad u(x, 0) = v(x),$$

is bounded by $\|v\|_\infty$, then the method is stable.

c) For stable implementation of the explicit Euler method for $u_t = u_{xx}$ with $h = 0.1$, we can use $k = h/2$.

d) Consider the equation $u'' - u' = 0$. Then the scheme

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{u_{i+1} - u_i}{h} = 0$$

is second order accurate.

2) Consider the problem

$$u_t = 2u_x, \quad x \in \mathbb{R}, \quad t > 0,$$

$$u(\cdot, 0) = v, \quad x \in \mathbb{R}.$$

a) Construct the forward differencing (upwind) scheme.

b) Show that if $\lambda \leq 1/2$, $\lambda = k/h$, then

$$\|U^n\|_C \leq \|v\|_C.$$

3) Consider the Dirichlet problem

$$\begin{aligned} -\Delta u &= f, & (x, y) \in \Omega &= (0,1) \times (0,1), \\ u &= 0, & \text{on } \Gamma &:= \partial\Omega. \end{aligned}$$

Introduce the grid points $(x_i, y_j) = (ih, jh)$, $0 \leq i, j \leq M$.

- Construct the 5-point finite-difference scheme using the centered differencing for the second order derivative. Let U_{ij} denote the approximation of $u(x_i, y_j)$.
- Show that if $f_{ij} \leq 0$ in Ω , then the numerical solution attains its maximum on the boundary.
- Formulate the matrix form of the scheme for $h = 1/5$.
- Show that truncation error is

$$\tau_{ij} = \frac{h^2}{12} \left(\partial_x^4 u(\xi_i, y_j) + \partial_y^4 u(x_i, \eta_j) \right), \quad (\xi_i, \eta_j) \in (x_{i-1}, x_{i+1}) \times (y_{j-1}, y_{j+1}).$$

- Show that

$$\max_{ij} |U_{ij} - u_{ij}| \leq Ch^2 \max_{\Omega} \{ |\partial_x^4 u|, |\partial_y^4 u| \}.$$

4) Consider the initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx} - u, & 0 < x < 1, & \quad t > 0. \\ u(x, 0) &= x(1-x), & 0 < x < 1, \\ u(0, t) &= u(1, t) = 0, & t > 0. \end{aligned}$$

Introduce the mesh-points $(x_j, t_n) = (jh, nk)$, $j = 0, 1, \dots, M$, $n = 0, 1, 2, \dots$

- Construct an explicit finite difference scheme.
- Write down the equations in matrix form.
- Find the solution at first time step when $h = 1/4$ and $k = 1/32$.
- Perform the von-Neumann stability analysis and show that the scheme is stable for

$$r < \frac{1}{2} - \frac{k}{4}, \quad \text{where } r = \frac{k}{h^2}.$$
