King Fahd University of Petroleum and Minerals
Department of Mathematics

MATH578: Applied Numerical Methods

Major Exam I, Term 241

Duration: 120 minutes, Marks: 25
Date: 10/19/2024

Name:____________________________ ID Number:____________________________

Instructions:

1. Solve all the problems and show all the steps.
2. No points for answers without justification.
3. Write legibly.
4. Computer, smart devices, and cell phone are not allowed during exam.

1. Use finite difference method with $h\,(\Delta x) = k\,(\Delta y) = 0.25$. Write the matrix (discrete) form of the following Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < 0.75, \quad 0 < y < 0.75,$$

subject to the boundary conditions

$$\begin{aligned} u\,(x,0) &= 0, \qquad u\,(x,0.75) = 0.75x, \quad 0 < x < 0.75, \\ u\,(0,y) &= 0, \qquad u\,(0.75,y) = 0.75y, \quad 0 < y < 0.75. \end{aligned}$$

Q2. Use $h(\Delta x) = k\,(\Delta t) = 0.25$, write the matrix (discrete) form

$$Aw^j = w^{j-1}$$

of the parabolic equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial^2 x}(x,t) = 0, \qquad 0 < x < 1, \quad t > 0;$$

subject to the initial and boundary conditions

$$\begin{aligned} u(x,0) &= \sin \pi x, \qquad 0 < x < 1, \\ u(0,t) &= u(1,t) = 0, \qquad 0 < t. \end{aligned}$$

using backward finite difference method. How many iteration are required to simulate the problem for time $T = 2$.

Q3. Using Finite Difference Method, approximate the solution to the wave equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, && 0 < x < 1, \quad 0 < t, \\
&u(0,t) = u(1,t) = 0, && 0 < t, \\
&u(x,0) = \sin \pi x, && 0 \le x \le 1, \\
&\frac{\partial u(x,0)}{\partial t} = 0, && 0 \le x \le 1,
\end{aligned}$$

with $h(\Delta x) = 0.5$ and $k(\Delta t) = 0.1$. Find the solution $u(x,t)$ at $T = 0.2$. Compare your results at $t = 0.2$ to the actual solution $u(x,t) = (\cos \pi t)(\sin \pi x)$.

Q4. Define of the following finite element spaces.

(i) $C(\Omega)$ (ii) $C^{\infty}(\Omega)$ (iii) $L^2(\Omega)$ (iv) $H^1(\Omega)$ (v) $H_0^2(\Omega)$, where $\Omega = [0\ 1]$.

Q5. (a) Write the key components of the Galekin Finite Element Method for solving partial differential equations.

(b) Consider the following differential equation

$$-u''(x) + u(x) = f(x) \qquad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

Show that the weak form (varitional form) is

$$(u', \phi') + (u, \phi) = (f, \phi), \qquad \forall\, \phi(x) \in H_0^1(0,1)$$

where

$$\begin{aligned} (u, \phi) &= \int_0^1 u(x)\,\phi(x)\,dx, \quad \text{and} \\ H_0^1(0,1) &= \{\phi(x), \phi(0) = \phi(1) = 0, \int_0^1 \phi^2 dx < \infty\}. \end{aligned}$$

(c) What is the advantage of Finite Element Method Over Finite Difference Method.

Extra page 1

Extra page 2