King Fahd University of Petroleum and Minerals
Department of Mathematics

MATH 578: Applied Numerical Methods-II

Final Exam: Semester 241 (120 minutes)

- -

NAME: STUDENT ID:

Note: Use of electronic devices/smartphones are not allowed. Use both side of the exam sheet. A well-organized and tidy paper will enhance the clarity of your responses and contribute to a better overall grade points.

Total Points: 35

Date: 21 Dec 2024

Instructor: Abdullah Shah (*abdullah.shah.1@kfupm.edu*.sa)

Q1. Points (7): Use the Backward Finite Difference Method to find the approximate solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}, \qquad 0 < x < \pi, \quad t > 0;$$

subject to the condition

$$\begin{aligned} u(0,t) &= u(\pi,t) = 0, \qquad 0 < t; \\ u(x,0) &= \sin x, \qquad 0 < x < \pi. \end{aligned}$$

Use $h = \frac{\pi}{4}, k = 0.1$. Perform only one iteration and find the error at $t = 0.1$ if the exact solution is

$$u(x,t) = e^{-t} \sin x.$$

Q2. Points $(5+2)$:

(a) Write the key components of the Galerkin Finite element Method for the differential equations

$$\begin{aligned} -\Delta u &= f, \qquad \text{in } \Omega, \\ u &= 0, \qquad \text{in } \partial\Omega. \end{aligned}$$

Clearly explain the *Weak formulation*, *Finite element solution*, *Mesh and Linear Basis Function*, and the resulting *Linear System*.

(b) Give definition with example of the two FE solution spaces given as $H^1(\Omega)$ and $H_0^2(\Omega)$.

Q3. Points $(3+4)$: Consider an equation

$$u_t + x u_x = 0, \tag{1}$$

with initial condition

$$u(x,0) = e^{-x^2}, \tag{2}$$

(a) Find $u(x,t)$ using the Method of Characteristics.

(b) Consider the Riemann problem for the equation

$$u_t + f(u)_x = 0,$$

with $f(u) = \frac{1}{2}u^2$ and initial condition

$$u(x,0) = \begin{cases} 2 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad 0 < x < 2 \\ 0 & \text{if} \quad x > 2 \end{cases}$$

Find the solution $u(x,t)$ at $t = 1$. Illustrate your solution graphically.

Q4. Points $(3+3+3)$:

(a) With explicit Euler method to approximate the time derivative, write the key components of the Finite Volume Method (FVM) for the equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0.$$

(b) Use Lax-Friedrichs numerical Flux at cell interfaces $i+\frac{1}{2}$ and $i-\frac{1}{2}$ as

$$F^n_{i+\frac{1}{2}} = \frac{1}{2}\left[f(u^n_i) + f(u^n_{i+1})\right] - \frac{\Delta t}{2\Delta x}\left(u^n_{i+1} - u^n_i\right).$$

and

$$F^n_{i-\frac{1}{2}} = \frac{1}{2}\left[f(u^n_{i-1}) + f(u^n_i)\right] - \frac{\Delta t}{2\Delta x}\left(u^n_i - u^n_{i-1}\right).$$

where $f(u) = au$, a is a constant. Derive Lax-Friedrichs Scheme for the equation given in part (a).

(c) Use the scheme in part (b) to solve

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,$$

for $t = 0.2$ with initial condition $u(x,0) = \sin(2\pi x)$, $x \in [0,1]$ and periodic boundary conditions. $\Delta x = 0.25$, $\Delta t = 0.1$ and $a = -2$.

Q5. Points (5): Use finite difference method with $h(\Delta x) = k(\Delta y) = 0.25$ and write the matrix (discrete) form of the following Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x}{y} + \frac{y}{x}, \qquad 1.25 < x < 2, \quad 1.25 < y < 2,$$

subject to the boundary conditions

$$\begin{aligned} u(x,1) &= x\ln x, \qquad u(x,2) = x\ln\left(4x^2\right), \quad 1.25 < x < 2, \\ u(1,y) &= y\ln y, \qquad u(2,y) = 2y\ln(2y), \quad 1.25 < y < 2. \end{aligned}$$