# EXAM 1

## Duration: 120 minutes

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- Show your work.
- Use the space provided to answer the question. If the space is not enough, continue on the back of the page or use the blank papers at the end and make sure to clearly refer to it.
- There are empty pages attached to this exam booklet.

| Problem | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
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| 6 | |
| Total | /100 |

# Problem 1 (20 points)

Let $\Omega$ be a subset of $\mathbb{R}^n$. Show that $\Omega$ is convex if and only if it contains all convex combinations of its elements.

# Problem 2 (20 points)

(a) Let $\Omega$ be a nonempty convex subset of $\mathbb{R}^n$. Show that $x \in \text{ri}\Omega$ if and only if for every $y \in \Omega$ there exists a $\gamma > 1$ such that $x + (\gamma - 1)(x - y) \in \Omega$.

(b) Let $\Omega_1$ and $\Omega_2$ be nonempty convex sets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Show that

$$\text{ri}\,(\Omega_1 \times \Omega_2) = \text{ri}\Omega_1 \times \text{ri}\Omega_2.$$

# Problem 3 (20 points)

In the Euclidean space $\mathbb{R}^4$, consider the set $\Omega$ defined as

$$\Omega = \left\{(x_1, x_2, x_3, 1) \in \mathbb{R}^4 : 1 - |x_1| - |x_2| - |x_3| \geq 0\right\}$$

(a) Show that $\Omega$ is a nonempty convex set.

(b) Find (no proof is necessary)

(i) $\text{int}\Omega$ (ii) $\text{ri}\Omega$ (iii) $\text{aff}\Omega$ (iv) the linear space parallel to $\text{aff}\Omega$

# Problem 4 (10 points)

Prove that the function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined as

$$f(x) = \frac{1}{\eta(x)} + e^{x^T A x}$$

is convex, where $\eta$ is a concave function [i.e $-\eta$ is convex] with $\eta(x) > 0$ for all $x \in \mathbb{R}^n$ and $A$ is a positive semidefinite symmetric $n \times n$ matrix.

[Hint: show that the individual summand functions are convex]

# Problem 5 (20 points)

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *quasiconvex* if $f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$ for all $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$.

(a) Show that a function $f$ is quasiconvex if and only if for any $\alpha \in \mathbb{R}$ the level set $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is a convex set.

(b) Give an example of a quasiconvex function that is not convex.

# Problem 6 (10 points)

Let $\Omega$ be a nonempty convex subset of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable real-valued function over $\mathbb{R}^n$. Suppose that the following inequality holds

$$f(y) \geq f(x) + (y - x)^T \nabla f(x) \quad \text{for all} \quad x, y \in \Omega.$$

Show that $f$ is convex.