EXAM 2

Duration: 120 minutes

ID:	Solution
NAME:	KEY.

- Show your work.
- Use the space provided to answer the question. If the space is not enough, continue on the back of the page or use the blank papers at the end and make sure to clearly refer to it.
- There are empty pages attached to this exam booklet.

Problem	Score
1	
2	
3	
4	
5	
Total	/100

Problem 1 (20 points)

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex function attaining its minimum at some point $\bar{x} \in \text{dom} f$. Show that the sets $\left\{\text{epi} f,\ \mathbb{R}^n \times \{f(\bar{x})\}\right\}$ form an extremal system.

Let $\Omega_1 = \text{epi} f$, $\Omega_2 = \mathbb{R}^n \times \{f(\bar{x})\}$

Since $\bar{x} \in \text{dom} f$, then $(\bar{x}, f(\bar{x})) \in \Omega_1 \cap \Omega_2$

Now, let $\alpha > 0$ be any positive real number.

We claim that $\Omega_1 \cap (\Omega_2 - (0,\alpha)) = \phi$. To prove this claim, assume otherwise that $(x,\lambda) \in \Omega_1 \cap (\Omega_2 - (0,\alpha))$. Then

$(x,\lambda) \in \Omega_1 \Rightarrow f(x) \leq \lambda$ & $(x,\lambda) \in \Omega_2 - (0,\alpha) \Rightarrow x \in \mathbb{R}^n$ &

$\lambda = f(\bar{x}) - \alpha$. So $f(x) \leq f(\bar{x}) - \alpha \Rightarrow f(x) - f(\bar{x}) \leq -\alpha < 0$

i.e $f(x) < f(\bar{x})$. This is a contradiction; since

$f(\bar{x}) \leq f(x)$ for all $x \in \mathbb{R}^n$.

Problem 2 (20 points)

Let $\overline{x} \in \text{dom } f$, where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is an extended real-valued convex function.

(a) If f is continuous at $\overline{x}$, show that $\overline{x} \in \textbf{int}(\text{dom } f)$.

(b) If $\overline{x} \in \textbf{int}(\text{dom } f)$, show that $\partial^\infty f(\overline{x}) = \{0\}$.

(a) Assume that f is continuous at $\bar{x}$. Then for every $\epsilon > 0$, there is a $\delta > 0$ such that if $x \in \mathbb{B}(\bar{x}; \delta)$, then

$|f(x) - f(\bar{x})| < \epsilon \iff f(x) - f(\bar{x}) < \epsilon$

$\Rightarrow f(x) < f(\bar{x}) + \epsilon < \infty$ because $\bar{x} \in \text{dom} f$

$\therefore x \in \text{dom} f \quad \forall x \in \mathbb{B}(\bar{x}; \delta)$; which shows that

$\bar{x} \in \text{int}(\text{dom} f)$.

(b) Assume that $\bar{x} \in \text{int}(\text{dom} f)$, then $N(\bar{x}; \text{dom} f) = \{0\}$

but $\partial^\infty f(\bar{x}) = N(\bar{x}; \text{dom} f) = \{0\}$.

Problem 3 (20 points)

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an extended real-valued convex function and let $\bar{x} \in \operatorname{dom} f$ be a local minimizer of f.

(a) Show that f attains its global minimum at this point.

(b) Is the global minimizer $\bar{x}$ unique? why?

(a) Assume that $\bar{x}$ is a local minimizer of f. That is there is a $\delta > 0$ such that $f(x) \geq f(\bar{x})$, $\forall x \in \mathbb{B}(\bar{x};\delta)$

Now let $y \in \mathbb{R}^n$ & consider the sequence $\{y_k\}$ defined as $y_k = (1-\frac{1}{k})\bar{x} + \frac{1}{k}y$, $k \in \mathbb{N}$. Then $y_k \in \mathbb{B}(\bar{x};\delta)$ for large enough k. By the convexity of f, we have $f(\bar{x}) \leq f(y_k) \leq (1-\frac{1}{k})f(\bar{x}) + \frac{1}{k}f(y) \Rightarrow \frac{1}{k}f(\bar{x}) \leq \frac{1}{k}f(y)$

or $f(\bar{x}) \leq f(y)$. So $\bar{x}$ is a global minimizer.

(b) No, For example $\delta(\bar{x}; [-1,1])$, $x \in \mathbb{R}$

the set $[-1,1]$ is a set of minimizers

Problem 4 (20 points)

Consider the set $\Omega \subset \mathbb{R}^2$ defined as

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid 3x_1^2 + x_2^4 \leq 1\}$$

(a) Give a brief justification why Ω is convex.

(b) Find $N(\bar{x}, \Omega)$ where $\bar{x} = \left(\frac{\sqrt{5}}{4}, \frac{1}{2}\right)$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be as $f(x_1, x_2) = 3x_1^2 + x_2^4$. Then the set Ω is the level set Ω_1 for f.

(a) $\nabla f(x_1, x_2) = \begin{pmatrix} 6x_1 \\ 4x_2^3 \end{pmatrix} \Rightarrow \nabla^2 f(x_1, x_2) = \begin{pmatrix} 6 & 0 \\ 0 & 12x_2^2 \end{pmatrix}$ which is positive semidefinite of all $(x_1, x_2) \in \mathbb{R}^n$. Therefore f is convex and all its level sets are convex.

(b) (1) $f(\bar{x}) = 3\left(\frac{5}{16}\right) + \frac{1}{16} = 1$

(2) $\partial f(\bar{x}) = \{\nabla f(\bar{x})\} = \left\{\begin{pmatrix} \frac{3\sqrt{5}}{2} \\ \frac{1}{2} \end{pmatrix}\right\}$. $\therefore 0 \notin \partial f(\bar{x})$

(3) f is continuous at $\bar{x}$ being a polynomial.

So

$$N(\bar{x}; \Omega_1) = \mathbb{R}_+ \partial f(\bar{x}) = \left\{\lambda \begin{pmatrix} \frac{3\sqrt{5}}{2} \\ \frac{1}{2} \end{pmatrix} : \lambda \geq 0\right\}.$$

Problem 5 (20 points)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the convex function defined as

$$f(x_1, x_2) = \max\{|x_1|, |x_2|\}, \quad \text{for all} \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Find $\partial f(x)$ for all $x \in \mathbb{R}^2$. Then find

(a) $\partial f(0,0)$.

(b) $\partial f(1,1)$.

(c) $\partial f(0,1)$.

let $f_1(x_1,x_2) = |x_1|$, $f_2(x_1,x_2) = |x_2|$. Then

$$\partial f_1(x_1,x_2) = \begin{cases} e_1, & x_1 > 0 \\ -e_1, & x_1 < 0 \\ \lambda e_1 - (1-\lambda)e_1, & x_1 = 0, \quad \lambda \in [0,1] \end{cases}$$

where $e_1 = (1,0)$, $e_2 = (0,1)$

$$\partial f_2(x_1,x_2) = \begin{cases} e_2, & x_2 > 0 \\ -e_2, & x_2 < 0 \\ \lambda e_2 - (1-\lambda)e_2, & x_2 = 0 \end{cases}$$

So for $(x_1,x_2) \neq (0,0)$

$$\partial f(x_1,x_2) = co\{(\text{sign}(x_i))e_i \mid |x_i| = f(x_1,x_2),\ i \in \{1,2\}\}$$

& $\partial f(0,0) = co\{\pm e_1, \pm e_2\} \leftarrow$ (a)

(b) $\partial f(1,1) = co\{e_1, e_2\}$,

(c) $\partial f(0,1) = co\{e_2\}$