

EXAM 2

Duration: 120 minutes

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| ID: | Solution |
| NAME: | KEY. |

- Show your work.
- Use the space provided to answer the question. If the space is not enough, continue on the back of the page or use the blank papers at the end and make sure to clearly refer to it.
- There are empty pages attached to this exam booklet.

| Problem | Score |
|---------|-------|
| 1 | |
| 2 | |
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| Total | /100 |

Problem 1 (20 points)

Let $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a convex function attaining its minimum at some point $\bar{x} \in \text{dom } f$. Show that the sets $\{\text{epif}, \mathbb{R}^n \times \{f(\bar{x})\}\}$ form an extremal system.

$$\text{Let } S_1 = \text{epif}, \quad S_2 = \mathbb{R}^n \times \{f(\bar{x})\}$$

since $\bar{x} \in \text{dom } f$, then $(\bar{x}, f(\bar{x})) \in S_1 \cap S_2$

Now, let $\alpha > 0$ be any positive real number.

We claim that $S_1 \cap (S_2 - (0, \alpha)) = \emptyset$. To prove this

claim, assume otherwise that $(x, \lambda) \in S_1 \cap (S_2 - (0, \alpha))$. Then

$$(x, \lambda) \in S_1 \Rightarrow f(x) \leq \lambda \quad \& \quad (x, \lambda) \in S_2 - (0, \alpha) \Rightarrow x \in \mathbb{R}^n \& \lambda = f(\bar{x}) - \alpha. \quad \text{So} \quad f(x) \leq f(\bar{x}) - \alpha \Rightarrow f(x) - f(\bar{x}) \leq -\alpha < 0$$

i.e. $f(x) < f(\bar{x})$. This is a contradiction; since

$$f(\bar{x}) \leq f(x) \quad \text{for all } x \in \mathbb{R}^n.$$

$\{x_0\} \times \mathbb{R} = \Omega$, this is selected

$\Omega \cap \Omega \ni (x_0, t)$ with, $t \in \mathbb{R}$, since

known for which x_0 and t exist

left now: $\phi = (x_0) \cap \Omega$ such that we have

$\Omega \cap (x_0) \ni (x_0, t)$ such that $x_0 = x_0$, $t = t$

$\exists x \in (x_0) \cap \Omega \ni (x_0, t) \in \Omega \Leftrightarrow x \in (x_0)$

$\Rightarrow x \in (x_0) \cap \Omega \Leftrightarrow x = x_0 \Leftrightarrow x = x_0$

\Rightarrow the contradiction is given $\Omega \cap (x_0) \neq \emptyset$.

$\exists x \in \Omega \cap (x_0) \neq \emptyset$

Problem 2 (20 points)

Let $\bar{x} \in \text{dom } f$, where $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is an extended real-valued convex function.

- (a) If f is continuous at \bar{x} , show that $\bar{x} \in \text{int}(\text{dom } f)$.
- (b) If $\bar{x} \in \text{int}(\text{dom } f)$, show that $\partial^\infty f(\bar{x}) = \{0\}$.

① Assume that f is continuous at \bar{x} . Then for every $\epsilon > 0$, there is a $\delta > 0$ such that if $x \in B(\bar{x}; \delta)$, then $|f(x) - f(\bar{x})| < \epsilon \iff f(x) - f(\bar{x}) < \epsilon$
 $\Rightarrow f(x) < f(\bar{x}) + \epsilon < \infty$ because $\bar{x} \in \text{dom } f$
 $\therefore x \in \text{dom } f \quad \forall x \in B(\bar{x}; \delta)$; which shows that $\bar{x} \in \text{int}(\text{dom } f)$.

② Assume that $\bar{x} \in \text{int}(\text{dom } f)$, then $N(\bar{x}; \text{dom } f) = \{0\}$
but $\partial^\infty f(\bar{x}) = N(\bar{x}; \text{dom } f) = \{0\}$.

Problem 3 (20 points)

Let $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be an extended real-valued convex function and let $\bar{x} \in \text{dom } f$ be a local minimizer of f .

- (a) Show that f attains its global minimum at this point.
- (b) Is the global minimizer \bar{x} unique? why?

① Assume that \bar{x} is a local minimizer of f . That is there is a $\delta > 0$ such that $f(x) \geq f(\bar{x})$, $\forall x \in B(\bar{x}; \delta)$. Now let $y \in \mathbb{R}^n$ & consider the sequence $\{y_k\}$ defined as $y_k = (1 - \frac{1}{k})\bar{x} + \frac{1}{k}y$, $k \in \mathbb{N}$. Then $y_k \in B(\bar{x}; \delta)$ for large enough k . By the convexity of f , we have $f(\bar{x}) \leq f(y_k) \leq (1 - \frac{1}{k})f(\bar{x}) + \frac{1}{k}f(y) \Rightarrow \frac{1}{k}f(\bar{x}) \leq \frac{1}{k}f(y)$ or $f(\bar{x}) \leq f(y)$. So \bar{x} is a global minimizer.

② No, For example $S(\bar{x}; [-1, 1])$, $x \in \mathbb{R}$
the set $[-1, 1]$ is a set of minimizers

exist. t to minimize loss σ in \mathcal{X} Test aspect ②

$(3;3)B \in \mathcal{X} A$, $(3)I \leq (x)I$ Test loss $\sigma(B) < \sigma$ in with
weights $\{w\}$ change all neurons \mathcal{S} to g to any

$$t: (3;3)B \rightarrow \text{test. } A \in \mathcal{X} \quad t: \frac{1}{N} \sum_{n=1}^N f_n^T + \frac{1}{N} (f_n - 1)^T = g$$

mean in t of changing all \mathcal{S} to g to all neurons equal
 $(\partial f_i / \partial w_j) \leftarrow (\partial f_i / \partial w_j) + (\partial f_i / \partial w_j) (f_j - 1) \geq (\partial f_i / \partial w_j) \geq (\partial f_i / \partial w_j)$

minimum loss σ in \mathcal{X} or $(\partial f_i / \partial w_j) = 0$

$\forall i \in \mathcal{X}, ((\partial f_i / \partial w_j) \neq 0)$ ignore with \mathcal{X} ③

minimum loss σ in \mathcal{X} for the all

Problem 4 (20 points)

Consider the set $\Omega \subset \mathbb{R}^2$ defined as

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid 3x_1^2 + x_2^4 \leq 1\}$$

(a) Give a brief justification why Ω is convex.

(b) Find $N(\bar{x}, \Omega)$ where $\bar{x} = \left(\frac{\sqrt{5}}{4}, \frac{1}{2}\right)$.

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be as $f(x_1, x_2) = 3x_1^2 + x_2^4$. Then the set Ω is the level set Ω_1 for f .

(a) $\nabla f(x_1, x_2) = \begin{pmatrix} 6x_1 \\ 4x_2^3 \end{pmatrix} \Rightarrow \tilde{\nabla} f(x_1, x_2) = \begin{pmatrix} 6 & 0 \\ 0 & 12x_2^2 \end{pmatrix}$ which is positive semidefinite of all $(x_1, x_2) \in \mathbb{R}^2$. Therefore f is convex and all its level sets are convex.

- (b) (1) $f(\bar{x}) = 3\left(\frac{\sqrt{5}}{4}\right)^2 + \frac{1}{16} = 1$
(2) $\partial f(\bar{x}) = \left\{ \nabla f(\bar{x}) \right\} = \left\{ \begin{pmatrix} \frac{3\sqrt{5}}{2} \\ \frac{1}{2} \end{pmatrix} \right\} \therefore 0 \notin \partial f(\bar{x})$
(3) f is continuous at \bar{x} being a polynomial.

So

$$N(\bar{x}; \Omega_1) = \mathbb{R}_+ \partial f(\bar{x}) = \left\{ \lambda \begin{pmatrix} \frac{3\sqrt{5}}{2} \\ \frac{1}{2} \end{pmatrix} : \lambda > 0 \right\}.$$

with next $x + \epsilon x = (x, x) +$ as $\epsilon \in \mathbb{R} : +$ the
next apply the law with $\alpha \in \mathbb{R}$ the
existing α divide $\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} = (x, x) + \Delta \in \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (x, x) + \Delta$ (2)

it has now $\alpha + \text{invert } \Delta \rightarrow (x, x)$ the following

$$(x) + \alpha + \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}^{-1} = \frac{1}{\alpha} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \epsilon = (x) + (0) \quad (3)$$

knowing a fixed Δ the equation $\alpha + \Delta =$

$$\left\{ \text{only} : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \right\} = (x) + \Delta = (x, x) \quad (4)$$

Problem 5 (20 points)

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the convex function defined as

$$f(x_1, x_2) = \max\{|x_1|, |x_2|\}, \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2.$$

Find $\partial f(x)$ for all $x \in \mathbb{R}^2$. Then find

(a) $\partial f(0,0)$.

(b) $\partial f(1,1)$.

(c) $\partial f(0,1)$.

$$\partial f_1(x_1, x_2) = \begin{cases} e_1, & x_1 > 0 \\ -e_1, & x_1 < 0 \\ \lambda e_1 + (1-\lambda)e_1, & x_1 = 0 \end{cases} \quad \text{where } e_1 = (1, 0) \\ e_2 = (0, 1) \quad \lambda \in [0, 1]$$

$$\partial f_2(x_1, x_2) = \begin{cases} e_2, & x_2 > 0 \\ -e_2, & x_2 < 0 \\ \lambda e_2 + (1-\lambda)e_2, & x_2 = 0 \end{cases}$$

So for $(x_1, x_2) \neq (0, 0)$

$$\partial f(x_1, x_2) = \text{co} \left\{ (\text{sign}(x_i))e_i \mid |x_i| = f(x_1, x_2), i \in \{1, 2\} \right\}$$

$$\text{and } \partial f(0,0) = \text{co} \{ \pm e_1, \pm e_2 \} \Leftarrow \underline{\text{(a)}}$$

$$\text{(b) } \partial f(1,1) = \text{co} \{ e_1, e_2 \}$$

$$\text{(c) } \partial f(0,1) = \text{co} \{ e_2 \}$$

$$\text{west. } l(x) = (x_1, x) + \quad l(x) = (x_1, x) + \begin{cases} \text{to} \\ \text{left} \end{cases}$$

$$(0, 1) \in \mathbb{P} \text{ with } \begin{cases} x_1 > 0 \\ x_1 < 0 \end{cases} \quad = (x_1, x) + \begin{cases} \text{up} \\ \text{down} \end{cases}$$

$$(1, 0) \in \mathbb{P} \text{ with } \begin{cases} x_1 > 0 \\ x_1 > 0 \end{cases} \quad = (x_1, x) + \begin{cases} \text{right} \\ \text{left} \end{cases}$$

$$(1, 1) \in \mathbb{P} \text{ with } \begin{cases} x_1 > 0 \\ x_1 > 0 \end{cases} \quad = (x_1, x) + \begin{cases} \text{up-right} \\ \text{up-left} \end{cases}$$

$$(0, -1) \in \mathbb{P} \text{ with } \begin{cases} x_1 < 0 \\ x_1 < 0 \end{cases} \quad = (x_1, x) + \begin{cases} \text{down} \\ \text{up} \end{cases}$$

$$(-1, 0) \in \mathbb{P} \text{ with } \begin{cases} x_1 < 0 \\ x_1 < 0 \end{cases} \quad = (x_1, x) + \begin{cases} \text{left} \\ \text{right} \end{cases}$$

$$(-1, -1) \in \mathbb{P} \text{ with } \begin{cases} x_1 < 0 \\ x_1 < 0 \end{cases} \quad = (x_1, x) + \begin{cases} \text{down-left} \\ \text{down-right} \end{cases}$$

$$\{0, 1, 2\} \text{ with } l(x) = (x_1, x) + \begin{cases} \text{up} \\ \text{down} \end{cases}$$

$$\{0, 1, 2\} \rightarrow \{0, 1, 2\} \times \{0, 1, 2\}$$

$$\{0, 1, 2\} \times \{0, 1, 2\} = \{0, 1, 2\}^2$$

$$\{0, 1, 2\}^2 = (0, 1, 2) + \{0, 1, 2\}$$

