FINAL EXAM

Duration: 180 minutes

ID:	
NAME:	

- Show your work.
- Use the space provided to answer the question. If the space is not enough, continue on the back of the page or use the blank papers at the end and make sure to clearly refer to it.
- There are empty pages attached to this exam booklet.

Problem	Score
1	
2	
3	
4	
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8	
Total	/140

Problem 1 (20 points)

Let $f : \mathbb{R}^n \to \overline{R}$ be a convex function and $f^* : \mathbb{R}^n \to [-\infty, \infty]$ be its *Fenchel Conjugate*.

(a) Prove that for any $\overline{x} \in \operatorname{dom} f$ we have

$$v \in \partial f(\overline{x}) \quad \text{if and only if} \quad f(\overline{x}) + f^*(v) = \langle v, \overline{x} \rangle.$$

(b) Show that $f_u^*(v) = f^*(v) + \langle u, v \rangle$, where $u \in \mathbb{R}^n$ and $f_u(x) = f(x - u)$.

(c) Find $f^*(v)$ where $f(x) = e^x$.

Problem 2 (20 points)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable $(C^2[\mathbb{R}^n])$.

(a) Prove that if the Hessian matrix $\nabla^2 f(\overline{x})$ is positive definite for all $\overline{x} \in \mathbb{R}^n$, then f is strictly convex.

(b) Give an example showing that the converse of part (a) is not true in general.

Problem 3 (20 points)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Use Farkas lemma to show that *exactly one* of the following systems has a solution

$$(I) \begin{cases} Ax = b, \\ x \geq 0 \end{cases} \qquad (II) \begin{cases} A^T y \leq 0, \\ b^T y > 0 \end{cases}$$

Problem 4 (20 points)

(a) Let F be a nonempty, closed, convex subset of $\mathbb{R}^n$ show that

$$F_\infty(x) = F_\infty(y) \quad \text{for any} \quad x, y \in F$$

(b) Find the horizon cone of the set $F = \{(x, y) \in \mathbb{R}^2 \mid y \geq |x|\}$.

Problem 5 (10 points)

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R},\ 1 \le i \le m$ be convex functions and consider the following optimization problem

$$\left.\begin{array}{ll} \text{min} & f(x) \\ \text{subjcet to} & g_i(x) \le 0, \quad 1 \le i \le m, \\ & x \in \Omega \end{array}\right\} \tag{P}$$

where Ω is a nonempty convex and closed subset of $\mathbb{R}^n$. Assuming that Slater's Constraint Qualification holds for (P), show that $\overline{x} \in \mathbb{R}^n$ is an optimal solution to (P) if and only if there are nonnegative Lagrange multipliers $\lambda_1, \cdots, \lambda_m$ such that

$$0 \quad \in \quad \partial f(\overline{x}) \quad + \sum_{i=1}^{m} \lambda_i \partial g_i(\overline{x}) \quad + N(\overline{x}; \Omega)$$

and $\lambda_i g_i(\overline{x}) = 0$ for all $1 \le i \le m$.

Problem 6 (10 points)

Consider the convex optimization problem

$$\min f(x), \qquad x \in \mathbb{R}^n \tag{$\mathcal{Q}$}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and Lipschitz continuous on $\mathbb{R}^n$ with Lipschitz constant $\mathscr{L} \geq 0$ and further the set S of optimal solutions to ($\mathcal{Q}$) is not empty. Given $x_1 \in \mathbb{R}^n$ and a sequence of numbers $\{\alpha_k\}, k \in \mathbb{N}$, consider the sequence $\{x_k\}$ generated by

$$x_{k+1} = x_k - \alpha_k v_k, \qquad v_k \in \partial f(x_k), \quad k \in \mathbb{N}$$

and show that for all $k \in \mathbb{N}$, we have

$$0 \leq \overline{f_k} - \overline{f} \leq \frac{d(x_1, S)^2 + \mathscr{L}^2 \sum_{i=1}^{k} \alpha_i^2}{2 \sum_{i=1}^{k} \alpha_i}$$

where

$$\overline{f_k} = \min\{f(x_1), \cdots, f(x_k)\}, \qquad \overline{f} = \min_{x \in \mathbb{R}^n} f(x).$$

Problem 7 (20 points)

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an extended real-valued function.

(a) Show that f is lower semicontinuous at $\bar{x} \in \mathbb{R}^n$ if and only if

$$f(\bar{x}) \leq \liminf_{k\to\infty} f(x_k)$$

for every sequence $\{x_k\}$ converging to $\bar{x}$.

(b) Is the function

$$f(x) = \begin{cases} x + \sin x, & x < 0 \\ \\ x^2 + 2, & x \geq 0 \end{cases}$$

lower semicontinuous? Justify your answer.

Problem 8 (20 points)

Consider the function

$$f(x) = \sum_{i=1}^{m} |x - i|, \qquad x \in \mathbb{R}$$

(a) Show that f is convex.

(b) Find $\partial f(x)$, for $x \in \mathbb{R}$.

(c) Solve the problem

$$\min_{x \in \mathbb{R}^n} f(x).$$