	College of Computing and Mathematics - Mathematics Department	
	Midterm Exam	INSTRUCTOR: DR. KAREEM ELGINDY
		2021 - Term 211
	Math 582: Nonlinear Programming	Duration: 120 minutes

Solution Manual

Answer ANY SIX QUESTIONS from the Following Seven Questions:

1. (a) **[8 marks]** Prove that the polyhedral set

$$\mathbb{S} = \left\{(x_1, x_2, x_3)^t \in \mathbb{R}^3 : x_1 + 5x_2 - 2x_3 \le 0, \quad 4x_1 - 7x_2 + 10x_3 \le -6\right\},$$

is a convex set.

Solution. We can write $\mathbb{S} = \left\{\boldsymbol{x} \in \mathbb{R}^3 : \mathbf{A}\boldsymbol{x} \le \boldsymbol{b}\right\}$, where $\boldsymbol{x} = (x_1, x_2, x_3)^t$, $\mathbf{A} = \begin{pmatrix} 1 & 5 & -2 \\ 4 & -7 & 10 \end{pmatrix}$, and $\boldsymbol{b} = (0, -6)^t$. 2pts To prove that $\mathbb{S}$ is a convex set, let $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \in \mathbb{S}$ and define $\boldsymbol{z} = \lambda \boldsymbol{x}^{(1)} + (1-\lambda)\boldsymbol{x}^{(2)}$, for some 2pts $\lambda \in [0, 1]$. Then $\mathbf{A}\boldsymbol{z} = \mathbf{A}\left[\lambda \boldsymbol{x}^{(1)} + (1-\lambda)\boldsymbol{x}^{(2)}\right] = \lambda \mathbf{A}\boldsymbol{x}^{(1)} + (1-\lambda)\mathbf{A}\boldsymbol{x}^{(2)} \le \lambda \boldsymbol{b} + (1-\lambda)\boldsymbol{b} = \boldsymbol{b}$. Hence, $z \in \mathbb{S}$ and $\mathbb{S}$ is convex. 2pts ◀

2pts

(b) **[7 marks]** Investigate the existence of a minimizing solution to the optimization problem

$$\begin{aligned} \min_{x,\, y,\, z} \quad & x^4 - e^{2y} + \cos\left(z^4\right) \\ \text{s.t.} \quad & x^6 + y^6 + z^6 \le 1. \end{aligned}$$

1pt

Solution. Notice that the feasible set $\mathbb{S} = \{(x, y, z)^t : x^6 + y^6 + z^6 \le 1\}$ is nonempty, since $(0, 0, 0)^t \in \mathbb{S}$. $\mathbb{S}$ is closed, since $\partial\mathbb{S} = \{(x, y, z)^t : x^6 + y^6 + z^6 = 1\} \subset \mathbb{S}$. 2pts Also, $\mathbb{S}$ is bounded, since

$$x^6 + y^6 + z^6 \le 1 \Rightarrow x, y, z \le 1 \Rightarrow \boldsymbol{x}^t\boldsymbol{x} = x^2 + y^2 + z^2 \le 3 < \infty. \quad \text{2pts}$$

Therefore, $\mathbb{S}$ is compact. Moreover, the function $f : \mathbb{R}^3 \to \mathbb{R}$, defined by $f(\boldsymbol{x}) = x^4 - e^{2y} + \cos\left(z^4\right)$, is continuous on $\mathbb{R}^3$, since it the sum of continuous functions. Hence, the conditions of the Weierstrass theorem are satisfied, and the problem admits at least one minimizing solution. 1pt ◀

1pt

2. Consider the following minimization problem:

$$\min_{x_1,\, x_2} \quad (x_1 - 1)^2 + (x_2 + 1)^2 \tag{1a}$$
$$\text{s.t.} \quad x_1 + x_2 \le 3, \tag{1b}$$
$$4x_1 + x_2 \le 5, \tag{1c}$$
$$x_1 \ge 0, \tag{1d}$$
$$x_2 \ge 0. \tag{1e}$$

(a) **[8 marks]** Write a necessary and sufficient condition for optimality of the problem.

Solution. Notice that the feasible set $\mathbb{S}$ can be defined as $\mathbb{S} = \{\boldsymbol{x} \in \mathbb{R}^2 : \mathbf{A}\boldsymbol{x} \le \boldsymbol{b}\}$, where $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\boldsymbol{b} = (3, 5, 0, 0)^t$. Thus, $\mathbb{S}$ is a polyhedral set, which is convex. 2pts Observe also that

$$\mathbf{H}(\bar{\boldsymbol{x}}) = \begin{bmatrix} \frac{\partial^2 f(\bar{\boldsymbol{x}})}{\partial x_1^2} & \frac{\partial^2 f(\bar{\boldsymbol{x}})}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(\bar{\boldsymbol{x}})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\bar{\boldsymbol{x}})}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \forall \bar{\boldsymbol{x}} \in \mathbb{S},$$

2pts 1pt

which is PD everywhere in $\mathbb{S}$. Therefore, f is strictly convex on $\mathbb{S}$ and the problem is a convex programming problem. 1pt Since, f is differentiable on $\mathbb{S}$, 1pt then a necessary and sufficient condition for $\bar{\boldsymbol{x}}$ to be an optimal solution for the problem is that $\left[\nabla f\left(\bar{\boldsymbol{x}}\right)\right]^t \left(\boldsymbol{x} - \bar{\boldsymbol{x}}\right) \ge 0, \quad \forall \boldsymbol{x} \in \mathbb{S}$. 1pt ◀

(b) **[7 marks]** Are $(1,0)^t$ and $(0,1)^t$ optimal solutions? Why?

Solution. At $\bar{\boldsymbol{x}} = (1,0)^t$ we have $\nabla f(\bar{\boldsymbol{x}}) = \left[\frac{\partial f(\bar{\boldsymbol{x}})}{\partial x_1}, \frac{\partial f(\bar{\boldsymbol{x}})}{\partial x_2}\right]^t$ 1pt $= [2(1-1), 2(0+1)]^t = [0,2]^t$. 1pt Therefore,

1pt 1pt

$$[\nabla f(\bar{\boldsymbol{x}})]^t (\boldsymbol{x} - \bar{\boldsymbol{x}}) = [0,2]\left[\boldsymbol{x} - (1,0)^t\right] = [0,2](x_1 - 1, x_2)^t = 2x_2 \overset{(1e)}{\geq} 0.$$

1pt

Therefore, $\bar{\boldsymbol{x}}$ satisfies the necessary and sufficient optimality condition; hence, it is the unique, strong global optimal solution for the optimization problem. 1pt Consequently, the point $\hat{\boldsymbol{x}} = (0,1)^t$ cannot be an optimal solution. 1pt ◀

3. (a) **[8 marks]** Check the quasiconvexity of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(\boldsymbol{x}) = x_1 + x_2^3$.

1pt 1pt 2pts 1pt

Solution. Let $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \in \mathbb{R}^2 : f\left(\boldsymbol{x}^{(1)}\right) \leq f\left(\boldsymbol{x}^{(2)}\right) \Rightarrow x_1^{(1)} + \left[x_2^{(1)}\right]^3 \leq x_1^{(2)} + \left[x_2^{(2)}\right]^3$. Also,

$\left[\nabla f\left(\boldsymbol{x}^{(2)}\right)\right]^t \left[\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}\right] = \left(1, 3\left[x_2^{(2)}\right]^2\right)\left[x_1^{(1)} - x_1^{(2)}, x_2^{(1)} - x_2^{(2)}\right]^t = \left[x_1^{(1)} - x_1^{(2)}\right] + 3\left[x_2^{(2)}\right]^2\left[x_2^{(1)} - x_2^{(2)}\right] \not\leq 0$, 1pt so f is not quasiconvex. For example, let $\boldsymbol{x}^{(1)} = (2,-2)^t, \boldsymbol{x}^{(2)} = (1,0)^t$. Note that $f\left(\boldsymbol{x}^{(1)}\right) = -6$ and $f\left(\boldsymbol{x}^{(2)}\right) = 1$, so that $f\left(\boldsymbol{x}^{(1)}\right) < f\left(\boldsymbol{x}^{(2)}\right)$. 1pt But $\left[\nabla f\left(\boldsymbol{x}^{(2)}\right)\right]^t \left[\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}\right] = (2-1) + 3(0)^2(-2-0) = 1 > 0$. 1pt ◀

(b) **[7 marks]** Which of the plots shown in Figure 1 below represent(s) the graph(s) of a function that is strictly quasiconvex but is neither quasiconvex nor convex. Why? *[Explain the reason only when the function satisfies the foregoing condition. Otherwise, state only the type(s) of the function from the three possible types, namely, strictly quasiconvex, quasiconvex, or convex.]*

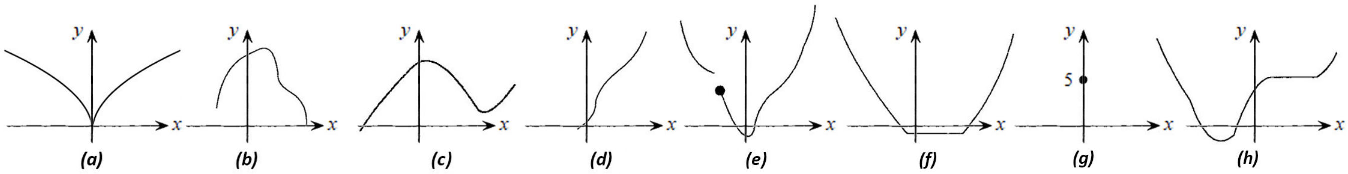

Figure 1: Plots of eight various functions.

1pt 1pt

Solution. Let $f_i, i = 1, \ldots, 8$ denote the function represented by the plots (a) through (h), respectively. Notice that f_1, f_4, and f_5 are strictly quasiconvex and quasiconvex but not convex. f_2 and f_3 are none of the three types. 1pt f_6 is strictly quasiconvex, quasiconvex, and convex. f_8 is quasiconvex but neither strictly quasiconvex nor convex. 1pt f_7 is the only function which is strictly quasiconvex but is neither quasiconvex nor convex. 1pt To illustrate further, notice that

$$f_7(\lambda x_1 + (1-\lambda)x_2) = 0 < 5 = f_7(0), \quad \forall x_1, x_2 \in \mathbb{S} : f_7(x_1) \neq f_7(x_2), \quad \lambda \in (0,1),$$

1pt

so f_7 is strictly quasiconvex. It is not quasiconvex because $f(x_1) = f(x_2) = 0$ for $x_1 = -1$ and $x_2 = 1$, but

$$f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) = f(0) = 5 > 0 = \max\{f(x_1), f(x_2)\}. \text{ ½pt}$$

It is also not convex, since for the same points x_1 and x_2,

$$f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) = f(0) = 5 > 0 = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2). \text{ ½pt}$$

◀

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ shown in Figure 2 below and defined by

$$f(x) = \min\{f_1(x), f_2(x)\},$$

where

$$f_1(x) = -|x|, \quad \forall x \in \mathbb{R}, \quad \text{and}$$
$$f_2(x) = 4 - (x-2)^2, \quad \forall x \in \mathbb{R}.$$

(a) **[10 marks]** Find the subgradients of f.

(b) **[5 marks]** Is f subdifferentiable at $x = 0, 1,$ and 5? Explain.

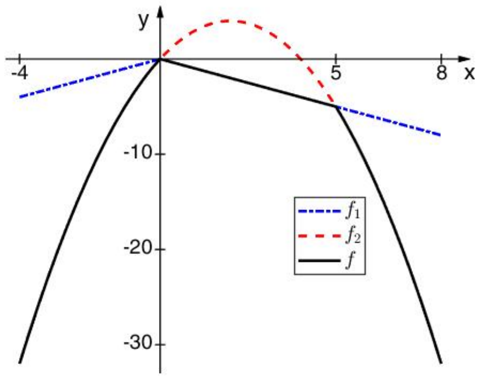

Figure 2: Plots of the functions $f, f_1,$ and f_2 on the interval $[-4, 8]$.

Solution. (a) Since $f_2(x) \geq f_1(x)$, for $0 \leq x \leq 5$, f can be redefined by

$$f(x) = \begin{cases} -x, & 0 \leq x \leq 5, \\ 4 - (x-2)^2, & \text{otherwise.} \end{cases}$$

Note that $\xi = -1$ is the unique subgradient of f at any point $x \in (0, 5)$. 2pts $\xi = -2(x-2)$ is the unique subgradient of f for $x < 0$ or $x > 5$. 2pts At $x = 0$ and 5, the subgradients are not unique because many supporting hyperplanes exist at each point. 2pts The family of subgradients at $x = 0$ and 5 are characterized in respective order by

$$\xi|_{x=0} = \lambda \nabla f_1(0) + (1-\lambda)\nabla f_2(0) = \lambda(-1) + (1-\lambda)(4) = 4 - 5\lambda, \quad \forall \lambda \in [0,1], \text{ 2pts}$$
$$\xi|_{x=5} = \lambda \nabla f_1(5) + (1-\lambda)\nabla f_2(5) = \lambda(-1) + (1-\lambda)(-6) = -6 + 5\lambda, \quad \forall \lambda \in [0,1]. \text{ 2pts}$$

(b) f is subdifferentiable at $x = 0, 1,$ and 5, since $\partial f(0) = [-1, 4]$, $\partial f(1) = \{-1\}$, and $\partial f(5) = [-6, -1]$.

2pts 1pt 2pts ◀

5. (a) **[10 marks]** Show that a polytope $\mathbb{S} \subseteq \mathbb{R}^n$ is a compact, convex set.

Solution. By definition, a polytope is the convex hull of a finite set of points. 2pts Let the finite set be $\mathbb{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$, for some $k \in \mathbb{R}^+$. Then, $\mathbb{X}$ is bounded and closed, 2pts since it is a finite union of the singleton sets $\{\boldsymbol{x}_i\}, i = 1, \ldots, k$, which are bounded and closed. 2pts Therefore, $\mathbb{X}$ is a compact set; 2pts consequently, $\mathbb{S} = \text{conv}(\mathbb{X})$ is compact. 2pts ◀

(b) **[5 marks]** Is $\text{conv}\left((-1,10)^t, (5,-1)^t, (0,0)^t, (-1/2,1)^t\right)$ a compact, convex set in $\mathbb{R}^2$? Explain.

Solution. Notice that $\text{conv}\left((-1,10)^t, (5,-1)^t, (0,0)^t, (-1/2,1)^t\right)$ is a polytope, 2pts since it is the convex hull of the finite set of points $\mathbb{S} = \left\{(-1,10)^t, (5,-1)^t, (0,0)^t, (-1/2,1)^t\right\}$. 2pts By Part (a) of the question, $\text{conv}\left((-1,10)^t, (5,-1)^t, (0,0)^t, (-1/2,1)^t\right)$ is a compact, convex set in $\mathbb{R}^2$. 1pt ◀

6. Let $\mathbb{S}$ be a nonempty, closed, convex set in $\mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^n - \mathbb{S}$.

(a) **[10 marks]** Show that there exists a nonzero vector $\boldsymbol{p}$ and a scalar α such that $\boldsymbol{p}^t\boldsymbol{y} > \alpha$ and $\boldsymbol{p}^t\boldsymbol{x} \leq \alpha$, for each $\boldsymbol{x} \in \mathbb{S}$.

Solution. By the given assumptions, there exists a unique point $\bar{\boldsymbol{x}} \in \mathbb{S}$ with minimum distance from $\boldsymbol{y}$. 1pt Furthermore, $\bar{\boldsymbol{x}}$ is the minimizing point if and only if $(\boldsymbol{y} - \bar{\boldsymbol{x}})^t(\boldsymbol{x} - \bar{\boldsymbol{x}}) \leq 0$, $\forall \boldsymbol{x} \in \mathbb{S}$. 1pt Now, let $\boldsymbol{p} = \boldsymbol{y} - \bar{\boldsymbol{x}}$ 1pt and $\alpha = \boldsymbol{p}^t\bar{\boldsymbol{x}}$. 1pt Notice that $\boldsymbol{p} \neq \boldsymbol{0}$ 1pt and

1pt

$$(\boldsymbol{y} - \bar{\boldsymbol{x}})^t(\boldsymbol{x} - \bar{\boldsymbol{x}}) = \boldsymbol{p}^t(\boldsymbol{x} - \bar{\boldsymbol{x}}) \leq 0, \quad \forall \boldsymbol{x} \in \mathbb{S} \Rightarrow \boldsymbol{p}^t\boldsymbol{x} \leq \boldsymbol{p}^t\bar{\boldsymbol{x}} = \alpha, \quad \forall \boldsymbol{x} \in \mathbb{S}.$$

1pt

Also, $\boldsymbol{p}^t\boldsymbol{y} - \alpha = \boldsymbol{p}^t\boldsymbol{y} - \boldsymbol{p}^t\bar{\boldsymbol{x}} = \boldsymbol{p}^t(\boldsymbol{y} - \bar{\boldsymbol{x}}) = \boldsymbol{p}^t\boldsymbol{p} = \|\boldsymbol{p}\|^2 > 0 \Rightarrow \boldsymbol{p}^t\boldsymbol{y} > \alpha, \quad \forall \boldsymbol{x} \in \mathbb{S}.$ ◀

1pt 1pt 1pt

(b) **[5 marks]** Show also that $\alpha = \max\{\boldsymbol{p}^t\boldsymbol{x} : \boldsymbol{x} \in \mathbb{S}\}$.

3pts

Solution. By the above answer, $\exists\, \bar{\boldsymbol{x}} \in \mathbb{S} : \boldsymbol{p}^t\boldsymbol{x} \leq \boldsymbol{p}^t\bar{\boldsymbol{x}} = \alpha \Rightarrow \alpha = \max\{\boldsymbol{p}^t\boldsymbol{x} : \boldsymbol{x} \in \mathbb{S}\}$. ← 2pts ◀

7. (a) **[8 marks]** Check the definiteness of the matrix $\mathbf{H} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ using The Superdiagonalization Algorithm.

1pt

Solution. Let $\mathbf{H} = (h_{ij})_{1 \leq i,j \leq 3}$. Clearly, $\mathbf{H}$ is symmetric since $\mathbf{H} = \mathbf{H}^t$. Notice that $h_{ii} > 0\, \forall i$, so by The Superdiagonalization Algorithm, we apply Gauss elimination to eliminate the off-diagonal elements of the first column:

1pt

$$\mathbf{H} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix} \xrightarrow[\text{elimination}]{\text{Gauss}} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

2pts

1pt

Since $h_{11} = 2 > 0$ and $\begin{pmatrix} 3/2 & -1 \\ -1 & 4 \end{pmatrix}$ is symmetric and PD, then $\mathbf{H}$ is PD. ← 1pt ◀

1pt 1pt

(b) **[7 marks]** Suppose that $f : \mathbb{S} \to \mathbb{R}$ is a twice differentiable function on a nonempty, open, convex set $\mathbb{S} \subseteq \mathbb{R}^n$. If the Hessian matrix of f is the matrix $\mathbf{H}$ given in Part (a) of the question, then show that f must be a convex function.

2pts 3pts

Solution. Since $\mathbf{H}(\bar{\boldsymbol{x}})$ is PD $\forall \bar{\boldsymbol{x}} \in \mathbb{S}$, then under the given assumptions, f must be strictly convex; hence, f is convex. ← 2pts ◀