	College of Computing and Mathematics - Mathematics Department	
	Final Exam	INSTRUCTOR: DR. KAREEM ELGINDY 2021/2022 - Term 211
	Math 582: Nonlinear Programming	Duration: 180 minutes

Answer ANY SIX QUESTIONS from the Following Eight Questions. Round ALL your calculations to at least three decimal digits.

1. (a) **[10 marks]** Let $\mathbb{S} = \{\boldsymbol{x} \in \mathbb{R}^2 : 0 \le x_1 \le \pi,\ 0 \le x_2 \le \sin x_1\}$ and $\boldsymbol{y} = (\pi, 1)^t$. Given that the root of the equation $t + (\cos t)\,[\sin t - 1] = \pi$ is about 2.663, find the minimum distance from $\boldsymbol{y}$ to $\mathbb{S}$, the unique minimizing point, and a separating hyperplane using the First Separation Theorem.

   (b) **[10 marks]** Are the first-order and second-order Taylor approximations of $f(x_1, x_2) = \ln(x_1 + 1) - 3x_2 + 5$ about the point $(0, 1)^t$ convex, concave, or neither? Explain.

2. Consider the following problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & 2e^{x_1} - 2x_1 x_2 \\ \text{s.t.} \quad & 3x_1 + 3x_2 - 6 = 0. \end{aligned}$$

   (a) **[10 marks]** Formulate a suitable penalty problem that can be solved using a second-order optimization method.

   (b) **[10 marks]** Perform one iteration of Newton's method for solving the penalty problem obtained in Part (a) using the penalty parameter $\mu = 10$ and starting from the point $(1, 1)^t$.

3. (a) **[10 marks]** Let $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m; j = 1, \ldots, l$ be continuous functions, and consider the following primal problem:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g_i(\boldsymbol{x}) \le 0\,, \quad i = 1, \ldots, m, \\ & h_j(\boldsymbol{x}) = 0\,, \quad j = 1, \ldots, l, \\ & \boldsymbol{x} \in \mathbb{X} \end{aligned}$$

   where $\mathbb{X} \in \mathbb{R}^n$ is a nonempty, compact set. Let θ be the Lagrangian dual function defined by $\theta(\boldsymbol{u}, \boldsymbol{v}) = \inf_{\boldsymbol{x} \in \mathbb{X}} \left\{ f(\boldsymbol{x}) + \sum_{i=1}^m u_i g_i(\boldsymbol{x}) + \sum_{j=1}^l v_j h_j(\boldsymbol{x}) \right\}$, where $\boldsymbol{u} = (u_1, \ldots, u_m)^t \ge \boldsymbol{0}$ and $\boldsymbol{v} = (v_1, \ldots, v_l)^t$. If $\bar{\boldsymbol{\xi}}$ is the subgradient in $\partial\theta(\boldsymbol{u}, \boldsymbol{v})$ having the smallest Euclidean norm, where $\partial\theta(\boldsymbol{u}, \boldsymbol{v})$ is the subdifferential of θ at $(\boldsymbol{u}, \boldsymbol{v})$, then show that the direction of steepest ascent of θ at $(\boldsymbol{u}, \boldsymbol{v})$ is given by

$$\bar{\boldsymbol{d}} = \begin{cases} \boldsymbol{0}, & \bar{\boldsymbol{\xi}} = \boldsymbol{0}, \\ \dfrac{\bar{\boldsymbol{\xi}}}{\|\bar{\boldsymbol{\xi}}\|}, & \bar{\boldsymbol{\xi}} \ne \boldsymbol{0}. \end{cases}$$

(b) **[10 marks]** Find and identify the minimum point(s) of the function

$$f(x_1, x_2) = 2x_1^3 + x_2^3 - 2x_1^2 + x_2^2 + 1.$$

4. (a) **[10 marks]** Let $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m; j = 1, \ldots, l$, and consider the following primal problem:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g_i(\boldsymbol{x}) \leq 0\,, \quad i = 1, \ldots, m, \\ & h_j(\boldsymbol{x}) = 0\,, \quad j = 1, \ldots, l, \\ & \boldsymbol{x} \in \mathbb{X} \end{aligned}$$

where $\mathbb{X}$ is a nonempty, open set in $\mathbb{R}^n$. Let $\boldsymbol{u} = (u_1, \ldots, u_m)^t$ and $\boldsymbol{v} = (v_1, \ldots, v_l)^t$, and define

$$\mathcal{X}(\boldsymbol{u}, \boldsymbol{v}) = \left\{ \bar{\boldsymbol{x}} : \bar{\boldsymbol{x}} = \arg\min_{\boldsymbol{x} \in \mathbb{X}} \left[f(\boldsymbol{x}) + \sum_{i=1}^{m} u_i g_i(\boldsymbol{x}) + \sum_{j=1}^{l} v_j h_j(\boldsymbol{x}) \right] \right\},$$

to be the set of optimal solutions to the Lagrangian dual subproblem. If $|g_i(\hat{\boldsymbol{x}})| \leq \epsilon$, $\forall i \in \mathcal{I}$, $g_i(\hat{\boldsymbol{x}}) \leq \epsilon\ \forall i \notin \mathcal{I}$, and $|h_j(\hat{\boldsymbol{x}})| \leq \epsilon$, $j = 1, \ldots, l$, at a point $\hat{\boldsymbol{x}} \in \mathcal{X}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}})$, where $\hat{\boldsymbol{u}} \in \mathbb{R}^m : \hat{\boldsymbol{u}} \geq \boldsymbol{0}, \hat{\boldsymbol{v}} \in \mathbb{R}^l, \mathcal{I} = \{i : u_i > 0\}$, and ϵ is a sufficiently small positive number, then show that $\hat{\boldsymbol{x}}$ is a near optimal primal solution.

(b) **[10 marks]** Perform two iterations of the steepest descent method for solving the minimization problem $\min_{x_1, x_2}(2x_1^2 + (x_2 - 1)^2 - 2x_1x_2)$ starting from the point $\boldsymbol{x}_1 = \boldsymbol{0}$.

5. Consider the following problem:

$$\begin{aligned} \max_{x_1, x_2} \quad & 2x_1^2 + x_2^2 - 2x_1x_2 \\ \text{s.t.} \quad & -2x_1 + 3x_2 \leq 6, \\ & 2x_1 + 4x_2 \geq 4, \\ & x_1, x_2 \geq 1, \\ & x_1, x_2 \leq 6. \end{aligned}$$

(a) **[10 marks]** Find the exact solution of the problem.

(b) **[10 marks]** Suppose we want to minimize the same objective function under the same constraints, then would there be a duality gap between the minimization problem and the Lagrangian dual problem obtained by letting the domain $\mathbb{X} = \{(x_1, x_2)^t : x_1, x_2 \geq 1,\ x_1, x_2 \leq 6\}$? Explain.

6. Consider the cutting plane algorithm for solving a Lagrangian dual problem of the following minimization problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & 2x_1^2 + x_2^2 \\ \text{s.t.} \quad & -2x_1 + 3x_2 \leq 6, \\ & x_1, x_2 \geq 1. \end{aligned}$$

(a) **[14 marks]** If the Lagrangian dual problem is obtained by letting the domain $\mathbb{X} = \{(x_1, x_2)^t : x_1, x_2 \geq 1\}$, perform two iterations of the cutting plane algorithm for solving the dual problem using the starting point $\boldsymbol{x}_0 = (2, 1)^t$.

(b) **[6 marks]** Are the approximations obtained in the second iteration optimal? Explain.

7. Consider the problem of minimizing $x_1 x_2$ subject to $x_1 + x_2 \geq 2$ and $x_2 \geq x_1$.

   (a) **[14 marks]** Find a point satisfying the Karush-Kuhn-Tucker conditions.

   (b) **[6 marks]** Is the point obtained in Part (a) an optimal solution.

8. (a) **[10 marks]** Consider the following problem:

$$\min_{x_1, x_2} \quad (3 - x_1 - x_2)^2 + (x_2^2 - x_1)$$

   Perform one complete iteration of Davidon-Fletcher-Powell method for solving the problem starting from the point $\boldsymbol{x}_1 = \boldsymbol{0}$.

   (b) Let $\boldsymbol{g}$ and $\boldsymbol{h}$ be the vector functions whose components are $g_1, \ldots, g_m$ and $h_1, \ldots, h_l$, respectively, and consider the following two problems:

$$\textbf{(Problem 1)} \quad \begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}, \\ & \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}, \\ & \boldsymbol{x} \in \mathbb{X} \end{aligned} \qquad \qquad \textbf{(Problem 2)} \quad \begin{aligned} & \sup \qquad \theta(\mu) \\ & \text{subject to } \mu \geq 0 \end{aligned}$$

   where $f, g_1, \ldots, g_m, h_1, \ldots, h_l$ are continuous functions defined on $\mathbb{R}^n, \mathbb{X} \subseteq \mathbb{R}^n$ is a nonempty set, $\theta(\mu) = \inf\{f(\boldsymbol{x}) + \mu\alpha(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{X}\}$ such that α is defined by

$$\alpha(\boldsymbol{x}) = \sum_{i=1}^{m} \phi[g_i(\boldsymbol{x})] + \sum_{i=1}^{l} \psi[h_i(\boldsymbol{x})],$$

   and ϕ and ψ are continuous functions satisfying the following:

$$\begin{aligned} \phi(y) = 0 \quad \text{if } y \leq 0 \quad \text{and} \quad \phi(y) > 0 \quad \text{if } y > 0, \\ \psi(y) = 0 \quad \text{if } y = 0 \quad \text{and} \quad \psi(y) > 0 \quad \text{if } y \neq 0. \end{aligned}$$

   Suppose also that $\exists\ \boldsymbol{x}_\mu \in \mathbb{X} : \quad \theta(\mu) = f(\boldsymbol{x}_\mu) + \mu\alpha(\boldsymbol{x}_\mu), \quad \forall \mu \geq 0$. Prove that the following two statements hold true:

   i. **[4 marks]** $\inf\{f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{X},\ \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0},\ \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}\} \geq \sup_{\mu \geq 0} \theta(\mu)$.

   ii. **[6 marks]** $f(\boldsymbol{x}_\mu)$ is a nondecreasing function of $\mu, \theta(\mu)$ is a nondecreasing function of μ, and $\alpha(\boldsymbol{x}_\mu)$ is a nonincreasing function of μ.