MAJOR EXAM 1

Duration: 120 minutes

ID:	
NAME:	

- Show your work.
- Use the space provided to answer the question. If the space is not enough, continue on the back of the page.

Problem	Score
1	/20
2	/15
3	/20
4	/15
5	/10
6	/10
7	/10
Total	/100
Score	/20

Problem 1 (20 points)

(a) Define the following terms:

- The *convex hull* of a set $S \subset \mathbb{R}^n$.
- A *simplex* in $\mathbb{R}^n$.

(b) State and prove Carathéodory's Theorem.

(c) Let $S = \{(0,0),(1,0),(0,1),(2,2)\} \subset \mathbb{R}^2$.

(i) Describe the convex hull $\text{conv}(S)$.

(ii) Is the point $(1,1)$ in $\text{conv}(S)$? If so, express it as a convex combination of points in S using at most three points.

(iii) Is the point $(2,1)$ in $\text{conv}(S)$? Justify your answer.

Problem 2 (15 points)

(a) Define a *supporting hyperplane* of a convex set S at a boundary point $x \in \partial S$.

(b) Let S_1 and S_2 be nonempty, disjoint convex sets in $\mathbb{R}^n$. Prove that there exist two nonzero vectors p_1 and p_2 such that $p_1^T x_1 + p_2^T x_2 \geq 0$ for all $x_1 \in S_1$ and $x_2 \in S_2$. (Hint: You may use separation theorems, but do not simply restate them.)

Problem 3 (20 points)

(a) Let $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 2x_1,\ x_1 \geq 0\}$.

(i) Show that C is a convex cone.

(ii) Find the explicit form of its polar cone C°.

(b) Consider the set $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\}$.

(i) List all extreme points and all extreme directions of S.

(ii) Express the point $(3, 1)$ as a convex combination of extreme points plus a nonnegative combination of extreme directions, or explain why this is not possible.

Problem 4 (15 points)

(a) Define a *subgradient* of a convex function at a point.

(b) Let $f(x) = \max\{2x, 1-x\}$ for $x \in \mathbb{R}$. Compute the set of all subgradients of f at $x_0 = 1/3$.

(c) For the function f in part (b), describe geometrically the supporting hyperplanes at $x_0 = 1/3$ and explain their relation to the subdifferential.

Problem 5 (10 points)

Let S be a nonempty open convex set in $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be differentiable on S. Prove that f is convex if and only if for each $x_1, x_2 \in S$ we have

$$[\nabla f(x_2) - \nabla f(x_1)]^T(x_2 - x_1) \geq 0.$$

Problem 6 (10 points)

Show that a function f on a convex set $S \subseteq \mathbb{R}^n$ is quasiconvex if and only if the function

$$\psi(t) = f(x + ty)$$

is quasiconvex on the interval

$$T = \{t \in \mathbb{R} \mid x + ty \in S\}$$

for every $x \in S$ and $y \in \mathbb{R}^n$.

Problem 7 (10 points)

Let $f : S \to \mathbb{R}$ be pseudoconvex on an open convex set $S \subset \mathbb{R}^n$. Prove that

$$\min_{x \in S} f(x) = f(x^*) \quad \iff \quad (x - x^*)^\top \nabla f(x^*) \geq 0, \qquad \forall x \in S.$$