MAJOR EXAM 2

Duration: 120 minutes

ID:	
NAME:	

- Show your work.
- Use the space provided to answer the question. If the space is not enough, continue on the back of the page.

Problem	Score
1	/20
2	/20
3	/25
4	/25
5	/10
Total	/100
Score	/20

Problem 1 (20 points)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable at $\overline{\mathbf{x}}$. Suppose that $\mathbf{d} \in \mathbb{R}^n$ is such that $\nabla f(\overline{\mathbf{x}})^t \mathbf{d} < 0$.

(a) Show that there exists a $\overline{\lambda} > 0$ such that $f(\overline{\mathbf{x}} + \lambda \mathbf{d}) < f(\overline{\mathbf{x}})$ for each $\lambda \in (0, \overline{\lambda})$.

(b) Show that if $\overline{\mathbf{x}}$ is a local minimum, then $\nabla f(\overline{\mathbf{x}}) = \mathbf{0}$.

(c) Give a counter example to (b) with $f : \mathbb{R}^2 \to \mathbb{R}$.

Problem 2 (20 points)

Let X be a nonempty open set in R^n and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1, \ldots, m\}$. Define the sets

$$\overbrace{S = \{\mathbf{x} \in X : \; g_i(\mathbf{x}) \leq 0 \quad \text{for } i \in \{1, \ldots, m\}\}}^{\text{feasible set}} \quad \text{and} \quad \underbrace{I = \{i : g_i(\overline{\mathbf{x}}) = 0\}}_{\text{index set for active constraints}}$$

<u>Let $\overline{\mathbf{x}} \in S$</u> and assume that

(1) $\{g_i\}_{i \in I}$ are differentiable at $\overline{\mathbf{x}}$
(2) $\{g_i\}_{i \notin I}$ are continuous at $\overline{\mathbf{x}}$,
(3) $\{g_i\}_{i \in I}$ are strictly pseudoconvex at $\overline{\mathbf{x}}$

Show that

$$G_0 = D,$$

where

$$\begin{aligned} G_0 &= \left\{\mathbf{d} : \nabla g_i(\overline{\mathbf{x}})^t \mathbf{d} < 0 \text{ for each } i \in I\right\} \\ D &= \left\{\mathbf{d} : \mathbf{d} \neq \mathbf{0}, \text{ and } \overline{\mathbf{x}} + \lambda \mathbf{d} \in S \text{ for all } \lambda \in (0, \delta) \text{ for some } \delta > 0\right\}. \end{aligned}$$

Problem 3 (25 points)

Consider the following nonlinear problem

$$\min \left(x_1 - \frac{9}{4}\right)^2 + (x_2 - 2)^2$$

subject to

$$\begin{aligned} x_2 - x_1^2 &\geq 0 \\ x_1 + x_2 &\leq 6 \\ x_1, \quad x_2 &\geq 0. \end{aligned}$$

1. Write the KKT optimality conditions and verify that these conditions hold true at the point $\overline{\mathbf{x}} = \left(\frac{3}{2}, \frac{9}{4}\right)^t$.
2. Interpret the KKT conditions at $\overline{\mathbf{x}}$ graphically.
3. Show that $\overline{\mathbf{x}}$ is indeed the unique global optimal solution.

Problem 4 (25 points)

Consider the nonlinear problem

$$\begin{array}{lrcl} \min & x_1^2 + x_2^2 & & \\ \text{subject to} & & & \\ & x_1 + x_2 - 4 & \geq & 0 \\ & x_1, x_2 & \geq & 0. \end{array}$$

(a) Verify that the optimal solution is $\bar{\mathbf{x}} = (2,2)^t$ with $f(\bar{\mathbf{x}}) = 8$.

(b) Let $X = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$, then write the Lagrangian dual problem.

(c) Verify that there is no duality gap for this problem.

(d) Starting with $\mathbf{x} = (3,3)^t$, perform one iteration of the cutting plane algorithm, check optimality and set-up the master program of the second iteration only (do not solve this in the second iteration).

Problem 5 (10 points)

Derive the Lagrangian dual of the linear program

$$\begin{array}{lll} \text{max} & c^t x & \\ \text{subject to} & & \\ & Ax & \leq b \end{array}$$