FINAL EXAM

Duration: 150 minutes

ID:	
NAME:	

- Show your work.
- Use the space provided to answer the question. If the space is not enough, continue on the back of the page.

Problem	Score
1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100
Score	/30

Problem 1 (20 points)

(a) Show that a set $K \subseteq \mathbb{R}^n$ is a convex cone if and only if it is closed under non-negative linear combinations. That is, for any $x, y \in K$ and any scalars $\alpha, \beta \geq 0$, prove that $\alpha x + \beta y \in K$.

(b) Let S be a nonempty set in $\mathbb{R}^n$ and let $\bar{\mathbf{x}} \in S$. Consider the set

$$C = \{\mathbf{y} : \mathbf{y} = \lambda(\mathbf{x} - \bar{\mathbf{x}}), \lambda \geq 0, \mathbf{x} \in S\}.$$

(i) Show that C is a cone and interpret it geometrically.

(ii) Show that C is convex if S is convex.

Problem 2 (20 points)

Let S be a nonempty convex set in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ be convex. Consider the perturbation function $\phi : \mathbb{R}^m \to \mathbb{R}$ defined as:

$$\phi(\mathbf{y}) = \inf\{f(\mathbf{x}) : \mathbf{g}(\mathbf{x}) \leq \mathbf{y}, \mathbf{x} \in S\}.$$

(a) Prove that ϕ is convex.

(b) Show that ϕ is monotone descreasing; that is

$$\mathbf{y}_1 \leq \mathbf{y}_2 \text{ (componentwise)} \Rightarrow \phi(\mathbf{y}_1) \geq \phi(\mathbf{y}_2).$$

Problem 3 (20 points)

Consider the following problem:

$$\begin{array}{lrcl} \min & -x_1 + x_2 & & \\ \text{subject to} & & & \\ & x_1^2 + x_2^2 - 2x_1 & = & 0 \\ & (x_1, x_2) & \in & \mathcal{X} \end{array}$$

where $\mathcal{X}$ is the convex combinations of the points $(-1, 0)$, $(0, 1)$, $(1, 0)$, and $(0, -1)$.

(a) Find the optimal solution graphically.

(b) Replace the set $\mathcal{X}$ by a suitable system of inequalities.

(c) Derive the **Karush–Kuhn–Tucker (KKT) conditions**.

Problem 4 (20 points)

Consider the following problem:

$$\begin{aligned}
\min\ & -2x_1 + 2x_2 + x_3 - 3x_4 \\
\text{subject to}\quad & x_1 + x_2 + x_3 + x_4 \leq 8 \\
& x_1 - 2x_3 + 4x_4 \leq 2 \\
& x_1 + x_2 \leq 8 \\
& x_3 + 2x_4 \leq 6 \\
& x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0.
\end{aligned}$$

Let $X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 \leq 8, x_3 + 2x_4 \leq 6; x_1, x_2, x_3, x_4 \geq 0\}$.

(a) Find the dual function θ explicitly.

(b) Write the dual problem.

(c) Verify that θ is differentiable at $(4, 0)$, and find $\nabla\theta(4, 0)$.

(d) Verify that $\nabla\theta(4, 0)$ is an infeasible direction for the dual problem, and find an improving feasible direction.

Problem 5 (20 points)

Let $\theta(\lambda) = 6e^{-2\lambda} + 2\lambda^2$.

(a) Show that θ has a unique minimum in the interval $(0, 1)$.

(b) Find the minimum of θ by the Golden section method (perform 2 iterations only).

(c) Using the Golden section, how many iterations are required to reach an uncertainty interval of length 0.001?

(d) Find the minimum of θ by Newton's method (perform 2 iterations only).