King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 601
Midterm Exam– Term 231
Monday, November 13, 2023
Allowed Time: 120 minutes

Instructor: **Dr. Boubaker Smii**

Name: ______________________________

ID #: ____________________

Section #: __________ Serial Number: __________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification !

Question #	Grade	Maximum Points
1		08
2		06
3		08
4		07
5		08
6		12
7		11
Total:		60

Exercise 1:(08)

A- Let X be a random variable with probability density function (pdf):

$$f(x) = x\, e^{-\frac{x^2}{2}}, \;\; 0 < x < \infty. \tag{a}$$

Find the probability density of $Y = X^2$.

B- Let X, Y be two random variables with joint probability density

$$f(x,y) = \begin{cases} \frac{1}{2}\, y\, e^{-xy}, & 0 \leq x \leq \infty,\; 0 < y < 2 \\ 0, & otherwise \end{cases} \tag{b}$$

Find $\mathbb{E}\Big[e^{\frac{X}{2}} \mid Y = 1\Big]$.

C- suppose that X_1 and X_2 are independent exponential random variables with respective means $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$, $\lambda_1, \lambda_2 > 0$. Find $P\{X_1 < X_2\}$.

<u>Exercise 2:</u> (06)

If X and Y are continuous random variables with corresponding probability densities and characteristic functions f_X, ϕ_X and f_Y, ϕ_Y respectively. Show that

$$\int_{\mathbb{R}} \phi_X(y)\, f_Y(y) e^{-ity}\, dy = \int_{\mathbb{R}} \phi_Y(x-t)\, f_X(x)\, dx \qquad \text{(c)}$$

Exercise 3:(08)

A-(05) Consider the Markov chain $\{X_n, n \geq 0\}$ with three states, $S = 1, 2, 3$, that has the following transition matrix

$$P = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.3 & 0.6 & 0.1 \end{pmatrix}$$

1. Draw the state transition diagram for this chain.

2. Determine $P(X_8 = 3, X_7 = 1, X_5 = 2 \mid X_3 = 2, X_2 = 1)$.

B-(03) A particle moves among states $0, 1$ and 2 according to a Markov process with the following transition probability matrix.

$$P = \begin{pmatrix} 0.6 & a & 0.1 \\ b & 0.3 & 0.4 \\ 0.3 & 0.2 & c \end{pmatrix}$$

where a, b and c are real numbers.

Let X_n be the position of the particle after the $n-$th move. Suppose that at the beginning, the particle is in state 1. Determine the probability $P[X_2 = k]$ where $k = 0, 1, 2$.

Exercise 4: (07)

Consider the geometric Brownian motion given by

$$X_t = e^{\mu t+\sigma B_t}, \ t \geq 0, \, \sigma > 0, \, \mu \in \mathbb{R}. \tag{d}$$

1- Explain why the classical integration fails to integrate the Brownian sample paths.

2- Give two major differences between the Riemann and Itô integrals.

3- Find $\mathbb{E}(X_t)$.

Exercise 5:(08)
Let $\{B_t,\, t \geq 0\}$ be a standard Brownian motion.
A- (03) Show whether or not $W_t = t\, B_{\frac{1}{t}}$, $t > 0$ and $W_0 = 0$ is also a Brownian motion.

B- (05) Given that $B_t^k = \int_0^t k\, B_s^{k-1}\, dB_s + \frac{k(k-1)}{2} \int_0^t B_s^{k-2}\, ds$, for $, k \geq 2$.
We define

$$X_k(t) := \mathbb{E}(B_t^k), \quad k = 2, 3, ...; \; t \geq 0. \tag{e}$$

1)- Prove that X_k, $k \geq 2$ is given in terms of a Riemann integral only.
(Give **explicitely** the expression of X_k .)

2)- Evaluate $\mathbb{E}(B_t^4)$ and $\mathbb{E}(B_t^6)$.
(**Hint:** You may use B)-1))

Exercise 6:(12)

Let $\{B_t,\, t \geq 0\}$ be a standard Brownian motion.

1-(03) Show that $\int_0^t s\, B_s\, dB_s = \alpha_t + a(t) + \int_0^t \gamma_s\, ds$, where α_t, γ_t are given stochastic processes and $a(t)$ a function of t.

2-(06)Evaluate the following integrals:

i)- $\int_0^t B_s^3\, dB_s$.

ii)- $\int_0^t (B_s^2 - s)\, dB_s$.

3-(03) Use the results in 1) and 2) to show that $\int_0^t (B_s^3 - 3\, s\, B_s)\, dB_s = \eta(t) + b(t)$, where $\eta(t)$ is a function of $B(t)$ and $b(t)$ is a function of t.

Exercise 7: (11)
Let X_t, Y_t be Itô processes in $\mathbb{R}$.
1)-(04) Prove that:

$$X_t\, Y_t = X_0\, Y_0 + \int_0^t Y_s\, dX_s + \int_0^t X_s\, dY_s + \int_0^t dX_s\, dY_s$$

2)-(07) Let $\Phi_t = \exp(-\alpha\, B_t + \frac{1}{2}\, \alpha^2\, t),\ \alpha \in \mathbb{R}$.
i)- Find $d\Phi_t$.
ii)- Given that: $dY_t = r\, dt + \alpha\, Y_t\, dB_t,\ r \in \mathbb{R}$. Prove that $Y_t = Y_0\, \Phi_t^{-1} + r\, \Phi_t^{-1} \int_0^t \Phi_s\, ds$
(**Hint**: Use 1)).